Anderson-Bernoulli localization on 3D lattice

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Schedule

1. Model definition and background
2. Framework of Bourgain-Kenig and Ding-Smart
3. Discrete unique continuation principle on $\mathbb{Z}^3$

Model definition and background

We study the operator $H := -\Delta + \delta V$ on $\mathbb{Z}^d$, where

- Δ is the discrete Laplacian:

$$\Delta u(a) = -2du(a) + \sum_{b \in \mathbb{Z}^d, |a-b|=1} u(b).$$

- $V : \mathbb{Z}^d \to \mathbb{R}$ is the Bernoulli random potential: $P[V(a) = 0] = P[V(a) = 1] = \frac{1}{2}$.
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Physics meaning:

An electron hopping inside a metal with uniform impurity.

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Definition (Anderson localization)

An operator H has **Anderson localization (AL)** in $I \subset sp(H)$, if for any polynomially bounded eigenfunction u with eigenvalue in I, there exist $c, C > 0$, such that $|u(a)| \leq C \exp(-c|a|), \forall a \in \mathbb{Z}^d$.

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Note that this is actually the **spectral localization** (to be distinguished from **dynamical localization**).

Theorem (Li and Z., 2019)

There exists $\lambda_* > 0$ *depending on* δ, *such that AL holds for* $H = -\Delta + \delta V$ *on* $\mathbb{Z}^3$, *in* $[0, \lambda_*]$.

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On $\mathbb{Z}$, AL holds in the whole spectrum with any nontrivial i.i.d. random potential and any $\delta > 0$.
(Kunz and Souillard, 1980)
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   The potential is defined as $V(x) = \sum_{j \in \mathbb{Z}^d} \epsilon_j \phi(x - j)$, where $\{\epsilon_j : j\}$ are i.i.d. Bernoulli random variables and ϕ is a nonnegative bump function supported in $\{x \in \mathbb{R} : |x| \leq \frac{1}{10}\}$.

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2. Inspired by a Liouville theorem of Buhovsky, Logunov, Malinnikova, and Sodin, on $\mathbb{Z}^2$ it was recently proved by Ding and Smart that for $-\Delta + \delta V$, AL holds in $[0, \varepsilon]$, for some $\varepsilon > 0$ (depending on δ).

Framework of Bourgain-Kenig and Ding-Smart

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For simplicity, take $\delta = 1$ and study $-\Delta + V$.

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Theorem (Wegner, 1981)

Take a self-adjoint operator A on $\mathbb{R}^n$, and $V = \mathrm{diag}(V_1, \cdots, V_n)$, an i.i.d. random potential with distribution density bounded by λ. Then for any $J \subset \mathbb{R}$,

$$\mathbb{P}(\text{exists an eigenvalue of } A + V \text{ in } J) \leq \lambda n |J|.$$

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Prove a weak Wegner-type estimate within induction on scales.

Arguments from the proof of the Wegner-type estimate:

- Let $Q_n = \{a \in \mathbb{Z}^d : \|a\|_\infty \leq n\}$.
- For $-\Delta + V$ on Q_n with Dirichlet boundary condition, let its eigenvalues be $\lambda_1 \leq \lambda_2 \leq \cdots$.

Wegner-Type Estimate

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- Given some fixed $r \in \mathbb{R}$ and j, we want a bound:

$$\mathbb{P}(|\lambda_j - r| < \exp(-n^{1-\varepsilon})) < n^{-\delta_0}$$

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  for some $\delta_0 > 0$.
- Consider the collection of potential:

$$\mathcal{A} := \{V : |\lambda_j - r| < \exp(-n^{1-\varepsilon})\} \subset \{0,1\}^{Q_n}.$$

  It is equivalent to show that $|\mathcal{A}| \leq 2^{(2n+1)^d} n^{-\delta_0}$.

Following Bourgain and Kenig, 2005, we wish to control $|\mathcal{A}|$ using variation arguments and Sperner's Theorem.

Theorem (Sperner's Theorem)

A family of sets is called a Sperner family, if none of them is a strict subset of another. If $\mathcal{M}$ is a Sperner family of subsets of $\{1, 2, \cdots, m\}$, then we have

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If $|\mathcal{A}| > \binom{m}{\lfloor \frac{m}{2} \rfloor}$ for $m = (2n+1)^d$, there are two different potentials $V_1 \leq V_2 \in \mathcal{A}$, such that for both $-\Delta + V_1$ and $-\Delta + V_2$, we have $|\lambda_j - r| < \exp(-n^{1-\varepsilon})$.
By a variation argument, this is not possible if $|u_j(a)|$ is not too small for some a with $V_1(a) \neq V_2(a)$.

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Theorem (Bourgain and Kenig, 2005, quantitative unique continuation principle on $\mathbb{R}^d$)

Suppose $u \in C^2(\mathbb{R}^d)$, $|\Delta u(a)| \leq C|u(a)| \leq C^2|u(\mathbf{0})|$ for any $a \in B_r$. Then

$$\int_{B_1(a)} |u(x)|dx \geq |u(\mathbf{0})|exp(-c|a|^{\frac{4}{3}}log(|a|))$$

for any $a \in B_{r/2}$.

Thus $|\mathcal{A}| \leq \binom{m}{\lfloor \frac{m}{2} \rfloor}$ for $m = (2n+1)^d$.

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Example: on $\mathbb{Z}^3$, let $u : (x, y, z) \mapsto (-1)^x \exp(sz)\mathbb{1}_{x=y}$, where $s \in \mathbb{R}_+$ is the constant satisfying $\exp(s) + \exp(-s) = 6$.
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Theorem (Ding and Smart, 2019)

Let $m' < m \in \mathbb{Z}_+$, and $\mathcal{M}$ be a family of subsets of $\{1, 2, \cdots, m\}$. Suppose $\mathcal{M}$ satisfies that, for every $A \in \mathcal{M}$, there is a set $B(A) \subset \{1, 2, \cdots, m\} \setminus A$ such that $|B(A)| \geq m'$, and $A \subset A' \in \mathcal{M}$ implies $A' \bigcap B(A) = \emptyset$. Then we have

$$|\mathcal{M}| \leq 2^m m^{\frac{1}{2}} m'^{-1}.$$

Wegner-Type Estimate: Generalized Sperner's Theorem

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Each potential $V \in \mathcal{A}$ corresponds to an $A_V \subset Q_n$, and we let $B(A_V) := \{a \in Q_n \setminus A_v : |u_j(a)| > C^{-n} \|u_j\|_{\ell^\infty(Q_n)}\} \subset Q_n \setminus A_V$.

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We wish to show that each $|B(A_V)| > n^{\frac{d}{2}+\delta_0}$.

The case where there is no potential:

Theorem (Buhovsky, Logunov, Malinnikova, and Sodin, 2017)

For $d = 2$, there exist universal constants $C, \varepsilon > 0$ such that the following holds. Suppose $u : Q_n \to \mathbb{R}$ satisfy $\Delta u = 0$ in Q_n and $|u(\mathbf{0})| = 1$, then

$$|\{a \in Q_n : |u(a)| \geq C^{-n}\}| \geq \varepsilon n^2.$$

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This is not true for arbitrary potential.
Consider $u : (x, y) \mapsto (-1)^x \mathbb{1}_{x=y}$, then we have $\Delta u = -4u$.

However, inspired by their method, a probabilistic version of 2D DUCP is proved.

Theorem (Ding and Smart, 2019)

There are constants $\alpha > 1 > \varepsilon > 0$ such that, if $\lambda \in \mathbb{R}$ and $n > \alpha$, then $\mathbb{P}(\mathcal{E}) \geq 1 - exp(-\varepsilon n^{\frac{1}{4}})$, where $\mathcal{E}$ denotes the event that

$$|\{a \in Q_n : |u(a)| \geq exp(-\alpha n log(n))|u(\mathbf{0})|\}| \geq \varepsilon n^{\frac{3}{2}} log(n)^{-1}$$

holds whenever $|\lambda - \lambda'| < \exp(-\alpha(n\log(n))^{\frac{1}{2}})$, and $(-\Delta + V)u = \lambda' u$ in Q_n.

Discrete unique continuation principle on $\mathbb{Z}^3$

Unlike the 2D lattice, on the 3D lattice, the desired DUCP holds for any potential, rather than just for typical ones.

Theorem (Li and Z., 2019)

There exists constant $p > \frac{3}{2}$ such that the following holds. For each $K > 0$, there are constants $C_0, C_1 > 0$, such that for any $n \in \mathbb{Z}_+$, and functions $u, V : \mathbb{Z}^3 \to \mathbb{R}$ with $\Delta u = Vu$, and $\|V\|_\infty \leq K$ in Q_n, we have that

$$|\{a \in Q_n : |u(a)| \geq \exp(-C_0 n)|u(\mathbf{0})|\}| \geq C_1 n^p.$$

Following the framework of Bourgain-Kenig and Ding-Smart, this implies 3D Anderson-Bernoulli localization.

We first prove a “small scale DUCP”.

Theorem (Li and Z., small scale DUCP)

For each $K > 0$, there exist C_0, C_1 relying only on K, such that for any $n \in \mathbb{Z}_+$ and functions $u, V : \mathbb{Z}^3 \to \mathbb{R}$ with $\Delta u = Vu$, and $\|V\|_\infty \leq K$ in Q_n, we have that

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Note that the power of n^2 cannot be improved, by the example $u : (x, y, z) \mapsto (-1)^x \mathbb{1}_{x=y}$.

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We consider four collections of planes in $\mathbb{R}^3$.

Definition

Let $\mathbf{e}_1 := (1,0,0)$, $\mathbf{e}_2 := (0,1,0)$, *and* $\mathbf{e}_3 := (0,0,1)$ *to be the standard basis of* $\mathbb{R}^3$, *and denote* $\lambda_1 := \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, $\lambda_2 := -\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, $\lambda_3 := \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$, $\lambda_4 := -\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$. *For any* $k \in \mathbb{Z}$, *and* $\tau \in \{1,2,3,4\}$, *denote* $\mathcal{P}_{\tau,k} := \left\{a \in \mathbb{R}^3 : a \cdot \lambda_\tau = k\right\}$.

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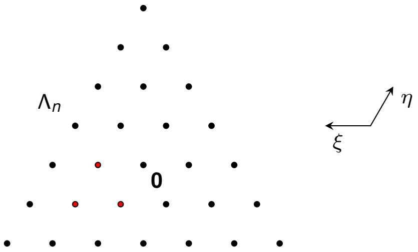
For any $k \in \mathbb{Z}$, *and* $\tau \in \{1,2,3,4\}$, *denote*
$\mathcal{P}_{\tau,k} := \left\{a \in \mathbb{R}^3 : a \cdot \lambda_\tau = k\right\}$.

We note that the intersection of $\mathbb{Z}^3$ with each of these planes is a **2D triangular lattice**.

Ideas of the Small Scale DUCP: 2D Triangular Lattice



Using arguments similar to that of Buhovsky, Logunov, Malinnikova, and Sodin, we get estimates on the 2D triangular lattice.

Theorem (Li and Z., 2D triangular lattice estimate)

There exist constants $C, c > 0$, such that for any positive integer n and any function $u : \Lambda \to \mathbb{R}$, if $|u(a) + u(a-\xi) + u(a+\eta)| < C^{-n}|u(\mathbf{0})|$ for any $a \in \Lambda_n$, then

$$\left|\{a \in \Lambda_n : |u(a)| > C^{-n}|u(\mathbf{0})|\}\right| > cn^2.$$

Ideas of the Small Scale DUCP: Decomposition

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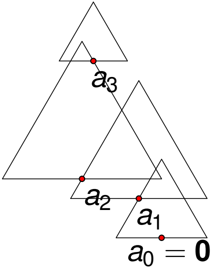
Step 1. On $\mathcal{P}_{1,0} = \{(x, y, z) : x + y + z = 0\}$, find a sequence of triangles $T_0, T_1, \cdots$.



For $a_0, a_1, \cdots$ being the middle points of one side of $T_0, T_1, \cdots$, we have $|u(a')| < C^{-n}|u(a_i)|$ for a' inside T_i.

Next we decompose $\mathbb{Z}^3$ into triangular lattice in $\mathcal{P}_{\tau,k}$.
Step 2. Using each T_i as basement, construct a **pyramid**.

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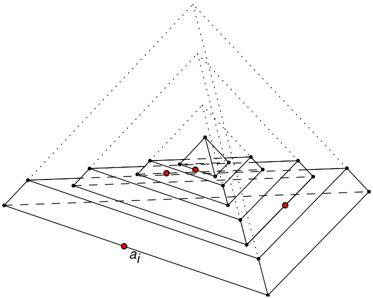

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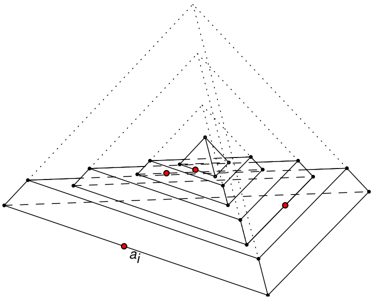

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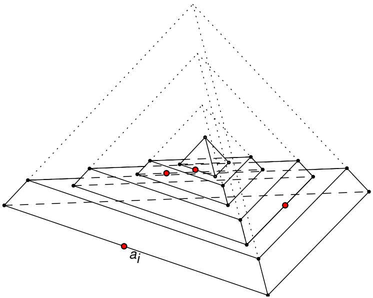

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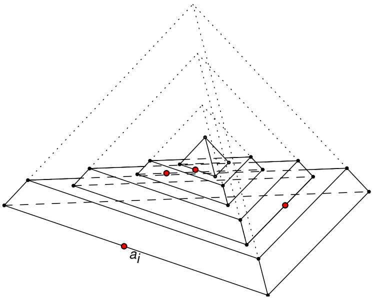

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- Apply the estimates on 2D triangular lattice to the faces.

3D Discrete Unique Continuation Principle

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Theorem (Large Scale DUCP)

There exist universal constants β and $\alpha > \frac{5}{4}$ such that for any positive integers $m \leq n$ and any positive real K, the following is true. For any $u, V : \mathbb{Z}^3 \to \mathbb{R}$ such that $\Delta u = Vu$ and $\|V\|_\infty \leq K$ in Q_n, we can find a subset $\Theta \subset Q_n$ with $|\Theta| \geq \beta \left(\frac{n}{m}\right)^\alpha$, such that

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Fix m and do induction on n:

find a few $b \in Q_n$, $|u(b)| \geq (K+11)^{-2n}|u(\mathbf{0})|$, and are far away from each other; then apply induction hypothesis to cubes centered at each b.

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For any $a \in \mathbb{Z}^3$, $s \in \{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3\}$, *we have*

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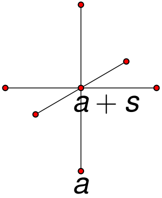
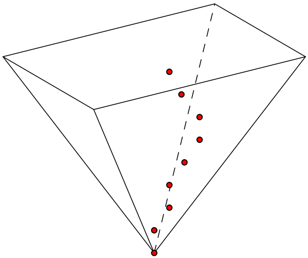
Keep walking in one of the $2d = 6$ directions, we can find a chain in a cone.





Thank you!

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