Mean Field Behavior during the Big Bang for Coalescing Random Walk

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Coalescing Random Walk (CRW) on a graph G:

- Initially one walker at each vertex of graph G.
- Each walker performs independent continuous time random walk. Jump rate equals 1 along each edge.
- Whenever two walkers meet(collide), they merge into one walker. This walker continues to perform random walk.

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- $r_{x,y} = \mathbf{1}[x \sim y]$ for general graph
- $r_{x,y} = \mathbf{1}[x \sim y]/d(x)$ for regular graph

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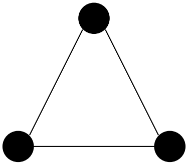
Motivation: duality with the voter model.

An example

Black=occupied, Green =vacant



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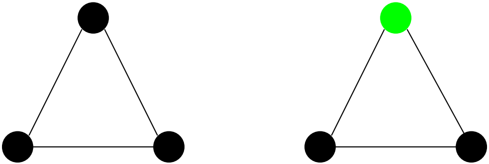

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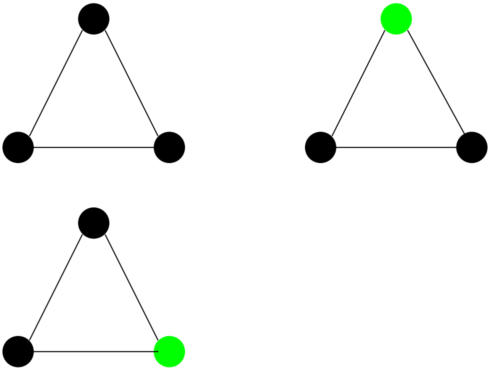

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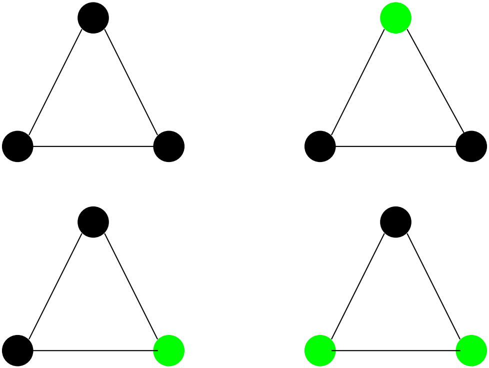

CRW on the complete graph

G is a complete graph (clique). $r_{x,y} = 1/(n-1)$.
L_t:# of walkers at time t.
$L_0 = n$. $L_t \to L_t - 1$ at rate $L_t(L_t - 1)/(n-1)$.

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$$\tau_{\mathrm{coal}} = \sum_{i=1}^{n} \frac{e_i}{i(i-1)/n}.$$

- $e_i, i \geq 1$ are i.i.d. with dist. Exp(1).
- $\frac{e_i}{i(i-1)/n}$ is the time it takes for the $n-i+1$-th coalescence to occur (corresponding to L_t from i to $i-1$).

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Related model: Kingman's coalescent. $L_0 = \infty$. $L_t \to L_t - 1$ at rate $L_t(L_t-1)/2$.

Decay of density on the complete graph

Define the expected density (expected fraction of occupied sites)

$$P_t = \frac{\mathbb{E}(L_t)}{n}.$$

Determine L_t: the time it takes to make h coalescences

$$\sum_{i=n-h+1}^{n} \frac{e_i}{i(i-1)/(n-1)} \sim n\left(\frac{1}{n-h} - \frac{1}{n}\right)$$

for $1 \ll h \ll n$. Set this expression to be t, we get

$$L_t = n - h \sim \frac{n}{t+1}.$$

Thus

$$P_t \sim \frac{1}{t+1}.$$

Spatial structure

Often there is a spatial structure.

- $\mathbb{Z}^d$.
- $\mathbb{T}^d$.
- General vertex transitive graphs.
- Random graphs (e.g., configuration model).

Heuristic argument [van den Berg-Kesten, 2000]

Consider $\mathbb{Z}^d$. $P_t = P_t(o)$: prob. that origin is occupied at time t.
Take $1 \ll \Delta(t) \ll t$.

$$-\frac{dP_t}{dt} = \mathbb{P}(o \text{ and } \mathbf{e}_1 \text{ occupied at } t)$$
$$\sim \sum_{x,y} \mathbb{P}(x \text{ and } y \text{ occupied at } t - \Delta(t)) \times$$
$$\mathbb{P}(x + S_{\Delta(t)} = o, y + S'_{\Delta(t)} = \mathbf{e}_1, x + S_r \neq y + S'_r, \forall r \leq \Delta(t))$$
$$\sim P^2_{t-\Delta(t)} \alpha_{\Delta(t)}.$$

- x and y are the location of the walkers that later come to o and $\mathbf{e}_1$. $S_\cdot, S'_\cdot$: independent random walks starting from o.
- $\alpha_{\Delta(t)}$: the probability that two time-reversed random walk starting from o and $\mathbf{e}_1$ don't collide by time $\Delta(t)$.

Results on $\mathbb{Z}^d$

Assuming $P_t \sim P_{t-\Delta(t)}$ and $\alpha_t \sim \alpha_{t-\Delta(t)}$. The heuristic suggests that $P_t \approx 1/(t\alpha_t)$ for moderately large t. This was known to be true for SRW on $\mathbb{Z}^d, d \geq 2$.

Theorem (Bramson-Griffeath, 1980)

Consider the CRW on $\mathbb{Z}^d$. Ww have, as $t \to \infty$,

$$P_t \sim \begin{cases} \frac{1}{\pi}\frac{\log t}{t} & d = 2 \\ (\gamma_d t)^{-1} & d \geq 3 \end{cases}$$

where γ_d is the probability that a simple random walkin $\mathbb{Z}^d$ starting from origin never returns to it.

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where γ_d is the probability that a simple random walk in $\mathbb{Z}^d$ starting from origin never returns to it.

By justifying previous heuristic argument, [van der Berg-Kesten, 2000] proved the same result for $d \geq 3$.

Approximation for coalescence time

π: stationary distribution.
Mean meeting time (the time it take for two indep. walkers to meet)

$$t_{\mathrm{meet}} = \mathbb{E}_{\pi^2}(\tau_{\mathrm{meet}}).$$

For complete graph $t_{\mathrm{meet}} = (n-1)/2$.

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Aldous and Fill conjectured that for finite transitive graph (transitivity means the graph looks the same from every vertex)

$$\frac{\tau_{\text{coal}}}{t_{\text{meet}}} \sim \sum_{i=2}^{\infty} \frac{e_i}{i(i-1)/2}.$$

Equality holds for complete graph (replacing ∞ by n). $e_i \sim Exp(1)$.

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The factor $i(i-1)/2$ counts the number of pairs

- Why exponential?

Aldous-Brown approximation

Lemma (Aldou-Brown, 1992)

For an irreducible reversible Markov chain on a finite state V with stationary distribution π and $A \subset V$, if we denote the hitting time of A by T_A and its density function w.r.t. the stationary chain by f_{T_A}, then

$$\left|\mathbb{P}_\pi(T_A > t) - \exp\left(-\frac{t}{\mathbb{E}_\pi(T_A)}\right)\right| \leq \frac{t_{\mathrm{rel}}}{\mathbb{E}_\pi(T_A)},$$

and

$$\frac{1}{\mathbb{E}_\pi(T_A)}\left(1 - \frac{2t_{\mathrm{rel}} + t}{\mathbb{E}_\pi(T_A)}\right) \leq f_{T_A}(t) \leq \frac{1}{\mathbb{E}_\pi(T_A)}\left(1 + \frac{t_{\mathrm{rel}}}{2t}\right).$$

Consider the product chain and take A to be the diagonal set. We have $E_\pi(T_A) = t_{\mathrm{meet}}$.

Second Prediction

[Oliveira, 2013] proved the Aldous-Fill conjecture under the condition $t_{\mathrm{mix}} \ll t_{\mathrm{meet}}$ (equivalent to $t_{\mathrm{rel}} \ll t_{\mathrm{meet}}$ due to Hermon). t_{mix} and t_{rel} quantify the rate of convergence to stationary distribution (See Markov Chains and Mixing Times).

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The time it takes to make h coalescences is about

$$t_{\mathrm{meet}} \sum_{i \geq n-h+1} \frac{e_i}{i(i-1)/2} \sim \frac{2t_{\mathrm{meet}}}{n-h}.$$

$$\frac{2t_{\mathrm{meet}}}{n-h} = t \Rightarrow n-h = \frac{2t_{\mathrm{meet}}}{t}.$$

Hence we have another prediction

$$P_t = \frac{E(L_t)}{n} = \frac{n-h}{n} \sim \frac{2t_{\mathrm{meet}}}{nt}.$$

Equivalence of the two predictions

Two predictions for P_t

- $$P_t \sim \frac{1}{t\alpha_t}$$
  where $\alpha_t = r(o)\mathbb{P}_{o,\nu_o}(\tau_{\text{meet}} > t)$ (ν_o is a random neighbor of o)
- $$P_t \sim \frac{2t_{\text{meet}}}{nt} \text{ for finite graphs}$$

They are equivalent to each other for many graphs by Kac's formula (in continuous time) and Aldous-Brown approximation:

$$\frac{1}{t_{\text{meet}}} \sim f_{T_A}(t) = \frac{2}{n}\mathbb{P}_{o,\nu_o}(\tau_{\text{meet}} > t) \text{ for } r(o) = 1.$$

Main Results: finite graphs

Theorem (Hermon-Li-Yao-Zhang, 2021)

Two predictions holds as long as $1 \ll t \ll t_{coal}$ (called the Big Bang regime since the number of particles is evolving rapidly in this regime) for

- *transitive graphs G_n such that diam$(G_n)^2 \ll n/\log n$,*
- *Configuration Model $\mathbb{CM}(n, D)$ with $3 \leq D < M$.*
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- *Configuration Model* $\mathbb{CM}(n, D)$ *with* $3 \leq D < M$.
  If D *is a constant* d *then* $\mathbb{CM}(n, D)$ *is random* d*-regular graph.*

Remarks:

- For such graphs t_{coal} and t_{meet} both have order n.
- By [Tessera and Tointon, 2019], $diam(G_n)^2 \ll n/\log n$ implies

$$\lim_{s\to\infty} \limsup_{n\to\infty} \sup_{x,y} \int_{s\wedge t_{\mathrm{rel}}}^{t_{\mathrm{rel}}} p_s(x,y)\mathrm{d}s = 0.$$

Configuration model

Construction of the configuration model $\mathbb{CM}_n(D)$

- Let D be a probability measure on $\mathbb{Z}_+$, and $n \in \mathbb{Z}_+$.
- We take n vertices labeled $1, \ldots, n$, and $d_1, \ldots, d_n$ i.i.d. sampled from D.
- For each vertex i we attach d_i half edges to it. Then we get G_n by uniformly matching all half edges, conditioned on $\sum_{i=1}^{n} d_i$ being even.

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The *local weak limit* $\mathbb{UGT}(D)$ of $\mathbb{CM}_n(D)$ is a unimodular Galton-Watson tree where

- the root has offspring distribution D
- later generations have offspring distribution D^*:

$$\mathbb{P}(D^* = k) := \frac{(k+1)\mathbb{P}(D = k+1)}{\sum_{i=0}^{\infty} i\mathbb{P}(D = i)}$$

Main Results: infinite Graphs

Theorem (Hermon-Li-Yao-Zhang, 2021)

The prediction $P_t(o) \sim 1/(t\alpha)$ *as* $t \to \infty$ *where*

$$\alpha = \mathbb{E}(r(o)\mathbb{P}_{o,\nu_o}(\tau_{meet} = \infty))$$

holds for

- *all transient transitive unimodular graphs, including*
  - *Cayley graphs*
  - *amenable graphs(=graphs with subexponential decay of return probability)*
- *unimodular Galton-Watson tree UGT(D). If D is a constant d then* $UGT(D) = \mathbb{T}^d$.

Duality with voter model

Voter model: at rate $r_{x,y}$, x adopts the opinion of y.
A site is occupied in CRW $\leftrightarrow$ the opinion is not lost in VM.

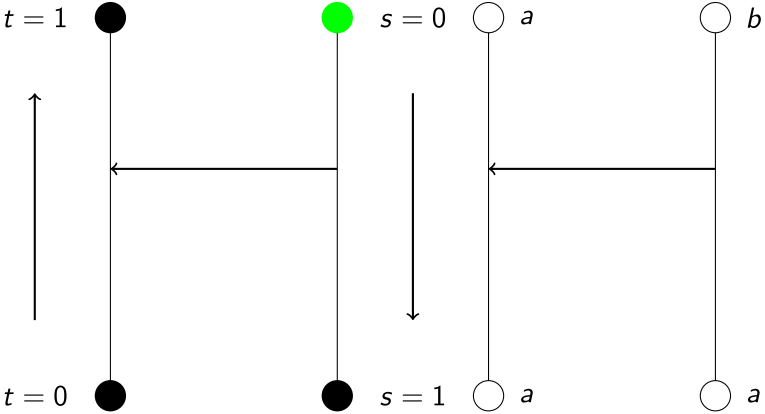

Figure: Left panel: CRW; right panel: voter model

Proof Sketch of [Bramson-Griffeath,1980]

n_t: # walkers that end up at origin at time t.
η_t: the voter model starting from different opinions at every site.
$\hat{N}_t := \{x : \eta_t(x) = \eta_t(o)\}$. [Kelly, 1977] gives

$$\mathbb{P}(\hat{N}_t = j) = j\mathbb{P}(n_t = j), j \geq 0, (\text{ i.e., size-biased verion of } n_t)$$

$$P_t = \mathbb{P}(n_t > 0) = \mathbb{E}(\hat{N}_t^{-1}) = \mathbb{E}\left[\left(\frac{\hat{N}_t}{\mathbb{E}(\hat{N}_t)}\right)^{-1}\right]\mathbb{E}(\hat{N}_t)^{-1}.$$

$\mathbb{E}(\hat{N}_t)$ is equal to $\mathbb{E}(R_{2t})$ where $R.$ is the range of a random walk.

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$\mathbb{E}(\hat{N}_t)$ is equal to $\mathbb{E}(R_{2t})$ where $R.$ is the range of a random walk.

Theorem (Sawyer, 1979)

Consider CRW on $\mathbb{Z}^d, d \geq 2$.

$$\lim_{t\to\infty} \mathbb{E}\left[\left(\frac{\hat{N}_t}{\mathbb{E}(\hat{N}_t)}\right)^k\right] = \frac{(k+1)!}{2^k}.$$

Proof Sketch of [Bramson-Griffeath,1980]-cont'd

A remark from [Bramson-Griffeath,1980]: Sawyer's theorem comes tantalizingly close to determining the asymptotics of P_t.
Gap: the function $f(x) = x^{-1}$ is unbounded near $x = 0$.

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Theorem (Bramson-Griffeath, 1980)

$$P_t = \begin{cases} O\left(\frac{\log t}{t}\right) & d = 2, \\ O\left(\frac{1}{t}\right) & d \geq 3. \end{cases}$$

Lemma (Bramson-Griffeath, 1980)

Sawyer's Theorem+upper bound on P_t gives the asymptotics of P_t.

Basically, the upper bound on P_t implies the $\hat{N}_t / E(\hat{N}_t)$ doesn't have too much mass near 0.

Transform to coalescence probability

Let N_t be the number of walkers that collide with the walker starting at U. $N_0 = 1$.

$$N_t = \sum_x \mathbf{1}[\text{the particle starting at } x \text{ coalesced with } U \text{ by time } t]$$

$$P_t = \mathbb{E}(N_t^{-1}) = [\mathbb{E}(N_t)]^{-1}\mathbb{E}\left[\left(\frac{N_t}{\mathbb{E}(N_t)}\right)^{-1}\right].$$

(A graph rooted at a uniform vertex is unimodular.)

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(A graph rooted at a uniform vertex is unimodular.)

$\mathbb{C}$: coalescence time for k+1 walkers.

$$\begin{aligned}\mathbb{E}(N_t^k) = \frac{1}{n}\sum_{x_1,\ldots,x_{k+1}\in V} & \mathbb{E}\left(\mathbf{1}[X_i(0) = x_i, \forall 1 \le i \le k+1]\right. \\ & \left.\times \mathbf{1}[\mathbb{C}(X_1,\ldots,X_{k+1}) \le t]\right) \\ & = n^k \mathbb{P}_{\pi^{\otimes k+1}}(\mathbb{C}(X_1,\ldots,X_{k+1}) \le t),\end{aligned}$$

Ingredients of the proof

Using the machinery in the proof of $\mathbb{Z}^d$ case by Braomson-Griffeath, it suffices to

- give an upper bound of P_t that differs from the 'true value' of P_t by a multiplicative constant,
- show that the coalescence probability

$$\mathbb{P}_{\pi^{k+1}}(\mathbb{C}(X_1, \ldots, X_{k+1}) \leq t) \sim (k+1)! \left(\frac{t}{t_{\mathrm{meet}}}\right)^k .$$

Another indication of mean field! B-G proof heavily relies on the specific geometric structure of $\mathbb{Z}^d$.

Solution

- For the first part, we show that for any $t > 0$,

$$c\frac{\inf_{x\in G}\int_0^t p_s(x,x)ds}{t} \leq P_t \leq C\frac{\sup_{x\in G}\int_0^t p_s(x,x)ds}{t}.$$

  where c and C are universal constants.
- For the second part, we use the reversibility of random walk to transform collision probability to non-colliding probability. If two forward paths collide at t then (after reversing time) the backward paths don't collide in $[0,t]$.

We want to estimate $\mathbb{P}_{\pi^{\otimes k+1}}(\mathbb{C}(X_1, \dots, X_{k+1}) \le t)$.

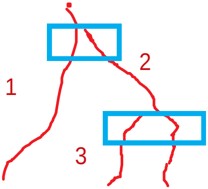
Consider the case $k = 1$. The probability that X_1 and X_2 collide within time interval $[t, t + dt]$ is about

$$
\begin{aligned}
&2 \sum_{u,v} \mathbb{P}(X_1(t) = u, X_2 \text{ jumps from } u \text{ to } v \text{ in } [t, t + dt]) \\
\sim &2 \sum_{u,v} \mathbb{P}(X_1(t) = u, X_2(t) = v, \text{ no collisions in } [0, t]) r_{v,u} dt \\
\sim &2 \sum_{u} \mathbb{P}(\gamma_1(0) = u) r(u) \times \\
&\quad \sum_{v} \frac{r_{u,v}}{r(u)} P(\gamma_2(0) = v) \mathbb{P}_{u,v}(\gamma_1(s) \neq \gamma_2(s), \forall 0 \le s \le t) dt,
\end{aligned}
$$

where γ_1 and γ_2 are the time-reversals of X_1, X_2 on $[0, t]$.

Collision Pattern and Branching Structure

We can imagine γ_1 is the parent of γ_2 and interpret the term $r_{u,v}/r(u)$ as the probability of the particle at u giving birth to a particle at location v.
Can be generalized to $k \geq 3$ paths.



If two walkers don't collide in time $O(t_{\mathrm{rel}})$, then they will also not collide before time t.

Lemma

For any $x \neq y$ and $0 < s < t$, the probability that two walkers starting from x and y collide between time s and t is bounded by

$$2\exp(-s/t_{\mathrm{rel}})\frac{\max_z \int_0^{2s} p_s(z,z)\mathrm{d}s}{\min_z \int_0^{2s} p_s(z,z)\mathrm{d}s} + \frac{8t}{n}(s^{-1} \vee r_{\max})$$

.

$r_{\max} = \max_x r(x)$. The error is small for $t_{\mathrm{rel}} \ll s \leq t \ll n$.

Open Question

For finite graphs our results (the expectation of the number of occupied sites) can be upgraded to a weak law of large numbers using negative correlation

$\mathbb{P}(\text{both } x \text{ and } y \text{ occupied at } t) \leq \mathbb{P}(x \text{ occupied at } t)\mathbb{P}(y \text{ occupied at } t).$

What about fluctuations? Do we have a Gaussian limit as in the mean field case ([Aldous, 1999])?

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Thanks!