# Shift-Invariance of the Colored TASEP

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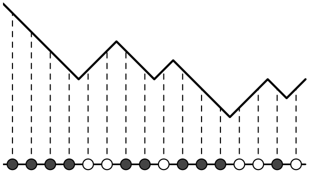
# The models: TASEP, colors, six-vertex

Totally Asymmetric Simple Exclusion Process (TASEP), and growing surface:



# TASEP and LPP

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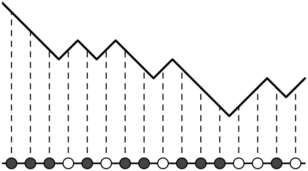

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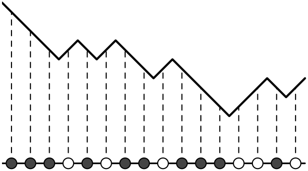

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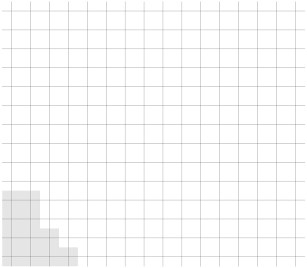

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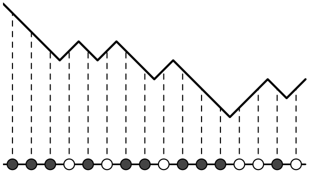

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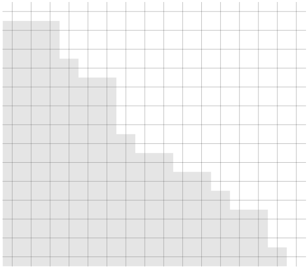
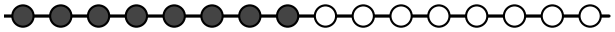
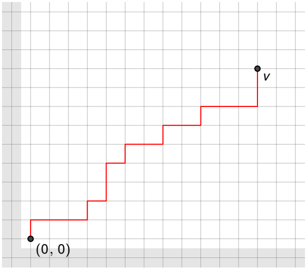


TASEP with step initial configuration also corresponds to Last Passage Percolation (LPP) with fixed starting point.

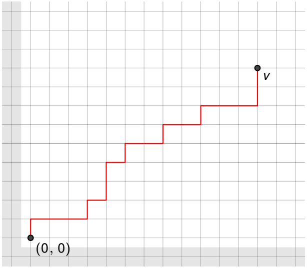





LPP on $\mathbb{Z}^2$:

- $\xi(v) \sim \mathrm{Exp}(1)$, i.i.d. $\forall v \in \mathbb{Z}^2$
- Passage time: $L_{u,v} := \max_\gamma \sum_{w \in \gamma} \xi(w)$

# Known Results on LPP/Corner growth




- $L_{(0,0),(n,n)} \sim 4n$ (Rost, 1981).
- $2^{-4/3}n^{-1/3}(L_{(0,0),(n,n)} - 4n)$ converges weakly to the GUE Tracy-Widom distribution (Johansson, 2000).
- Point to line profile (Borodin and Ferrari, 2008)

$$2^{-4/3}n^{-1/3}\left(L_{(0,0),(n-x(2n)^{2/3},n+x(2n)^{2/3})} - 4n\right) \Rightarrow \mathcal{A}_2(x) - x^2$$

  $\mathcal{A}_2$ is stationary and absolute continuous with respect to Brownian motion (Corwin and Hammond, 2014).
- KPZ fixed point (Matetski, Quastel, and Remenik, 2017) Airy sheet (Dauvergne, Ortmann, and Virág, 2018).

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One particle at each integer, and the particle at $i$ is labeled $i$.

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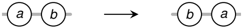


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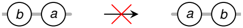

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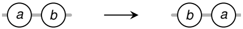


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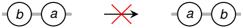
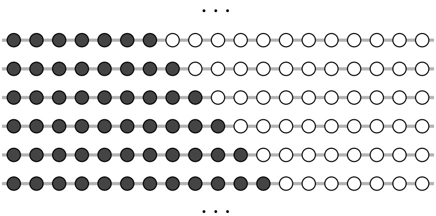
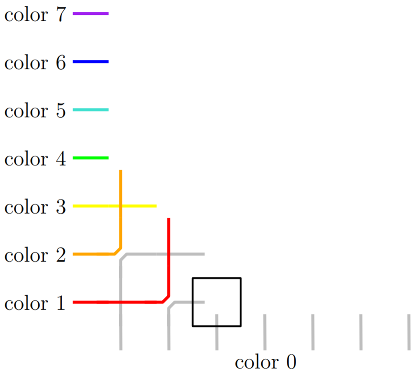


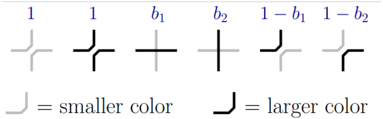

Alternative description: a family of coupled step initial TASEPs, by considering all particles $\leq i$.



# Stochastic Colored 6-Vertex Model: a discrete analogue

A general model in integrable probability (figures from Vadim):



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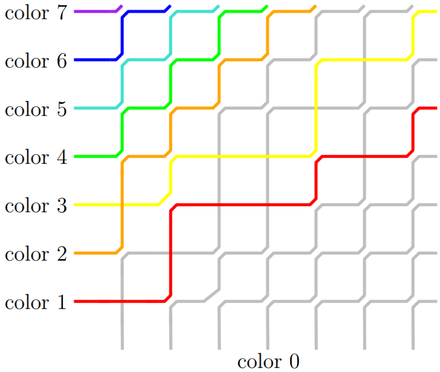
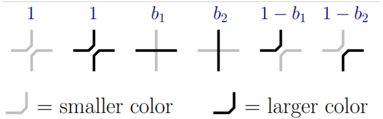


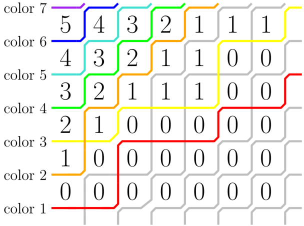
# Symmetry

Let $\zeta_t : \mathbb{Z} \to \mathbb{Z}$ be the configuration of the colored TASEP at time $t$. In particular, $\zeta_0$ is the identity map.

The following has the same distribution as $\zeta_t$:

- $x \mapsto \zeta_t(x - y) + y$ for any $y \in \mathbb{Z}$
- $x \mapsto -\zeta_t(-x)$
- $\zeta_t^{-1}$ (color-to-position symmetry, see e.g. Amir, Angel, and Valkó, 2011; Angel, Holroyd, and Romik, 2009; Borodin and Bufetov, 2021)
- New shift/flip invariance by Borodin, Gorin, and Wheeler, 2019; Galashin, 2020, from the colored stochastic 6-vertex model

Height function in the colored stochastic 6-vertex model (figure from Vadim).




$\mathcal{H}^{\geq i}(x, y)$: number of paths with color $\geq i$ to the right of/below $(x, y)$.

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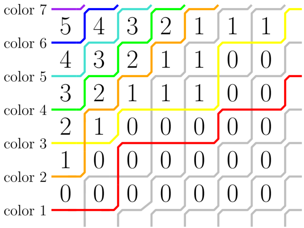


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**Theorem (Borodin, Gorin, and Wheeler, 2019)**

*Let $1 \leq \tau \leq n$, and $k_i' = k_i + \mathbb{1}[i = \tau], \mathcal{U}_i' = \mathcal{U}_i + (0, \mathbb{1}[i = \tau])$. Under intersection conditions, we have*

$$\left\{\mathcal{H}^{\geq k_i}(\mathcal{U}_i)\right\}_{i=1}^{n} \stackrel{d}{=} \left\{\mathcal{H}^{\geq k_i'}(\mathcal{U}_i')\right\}_{i=1}^{n}.$$

Extended by Galashin, 2020 and Dauvergne, 2020.

# New shift-invariance for colored TASEP

Passage times in colored TASEP:

$$T^A_{B,C} = \inf\{t \geq 0 : |\{x \geq A + B + 1 - C : \zeta_t(x) \leq A\}| \geq C\}.$$

Corresponds to: LPP time $L_{(1,1),(B,C)}$.
(recall: $\{x : \zeta_t(x) \leq A\}$ gives step initial TASEP)

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One can degenerate the results in Galashin, 2020 to the following:

**Theorem**

*Let $1 \leq \tau \leq n$ and $A^+_i = A_i + \mathbb{1}[i > \tau]$. Under intersection conditions,*

$$\max_i T^{A_i}_{B_i,C_i} \stackrel{d}{=} \max_i T^{A^+_i}_{B_i,C_i}.$$

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We get a stronger result for this.

**Theorem (Zhang, 2021)**

*Let* $1 \leq \tau \leq g$ *and* $A_{i,j}^+ = A_{i,j} + \mathbb{1}[i > \tau]$*. Under intersection conditions,*

$$\left\{\max_{1\leq j\leq k_i} T^{A_{i,j}}_{B_{i,j},C_{i,j}}\right\}_{i=1}^{g} \stackrel{d}{=} \left\{\max_{1\leq j\leq k_i} T^{A_{i,j}^+}_{B_{i,j},C_{i,j}}\right\}_{i=1}^{g}.$$

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The intersection conditions:

$$A_{i,j} \le A_{i',j'}, \quad A^+_{i,j} + B_{i,j} \ge A^+_{i',j'} + B_{i',j'}, \quad A^+_{i,j} - C_{i,j} \ge A^+_{i',j'} - C_{i',j'},$$

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For example: by using it repeatedly, for each $N$ we have

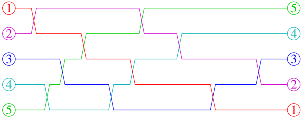
$$\left\{ T^{1}_{N-k,k} \right\}_{k=1}^{N-1} \overset{d}{=} \left\{ T^{k}_{N-k,k} \right\}_{k=1}^{N-1} .$$

Previously, only know that the maximum are equal in distribution.

# The Oriented Swap Process

# Sorting Network

A shortest path in the group $S_N$, from $(1, \cdots, N)$ to $(N, \cdots, 1)$, swapping two neighboring numbers at a time.

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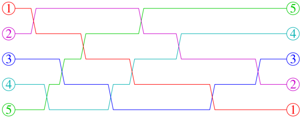


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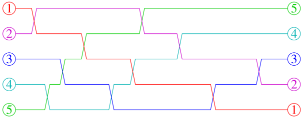


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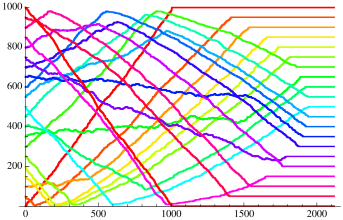
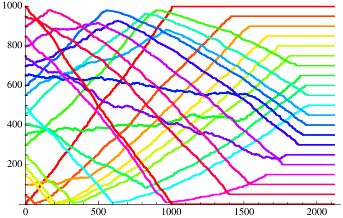



A simulation with $N = 1000$ (from Angel, Holroyd, and Romik, 2009).

# OSP and colored TASEP




OSP can be viewed as the colored TASEP on an interval $[1, N]$.

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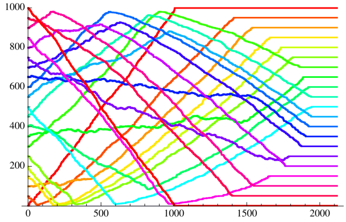


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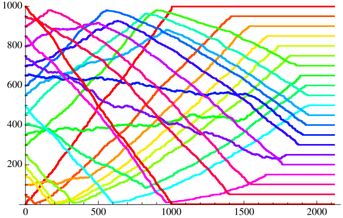


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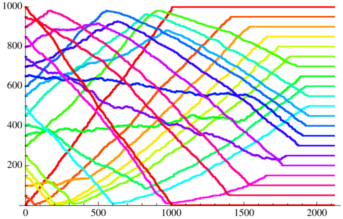
**Absorbing time**: the time when the OSP terminates.

## Question

*What are the fluctuations and limiting law of the absorbing time?*

# A conjecture on the finishing times




Take $\mathbf{U}_N = (U_N(1), \ldots, U_N(N-1))$, where $U_N(k)$ is the last time such that a swap happens between the sites $k$ and $k+1$.

**Conjecture (Bisi, Cunden, Gibbons, and Romik, 2020; Bufetov, Gorin, and Romik, 2020)**

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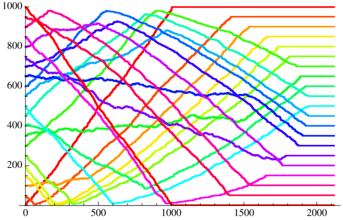


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Some results

1. Single $k$.
2. $N \leq 6$ (computer-assisted).
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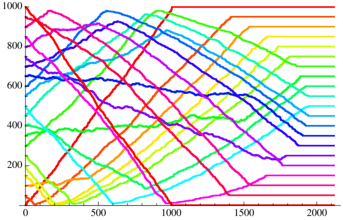


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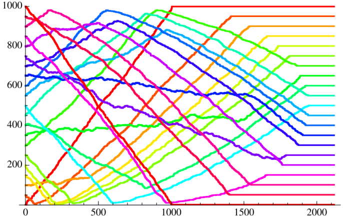
# Result on OSP and implications

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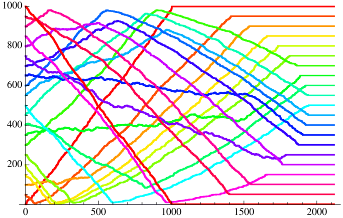


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Some implications (using the asymptotic results of LPP):

1. Under $N^{2/3}$, $N^{1/3}$ scaling, $\mathbf{U}_N$ converges to the parabolic Airy$_2$ process.
2. Consider $k_*$ such that the last swap is between sites $k_*$ and $k_* + 1$; then $N^{-2/3}(k_* - N/2)$ converges.
3. In scale smaller than $N^{2/3}$, $\mathbf{U}_N$ converges to simple random walk.

# Proof ideas

**From the colored TASEP shift invariance to OSP finishing times:**

$$\{L_{(1,1),(k,N-k)}\}_{k=1}^{N-1} \stackrel{d}{=} \{T^{1}_{N-k,k}\}_{k=1}^{N-1} \stackrel{d}{=} \{T^{k}_{N-k,k}\}_{k=1}^{N-1} \stackrel{d}{=} \mathbf{U}_N.$$

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## Shift invariance: an example

Take $B, C \geq 2$. Goal: show that $T^0_{B,1}, T^0_{1,C} \stackrel{d}{=} T^0_{B,1}, T^1_{1,C}$.

$T^0_{B,1}, T^0_{1,C}$: TASEP with labels $\leq 0$

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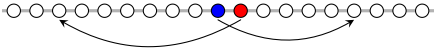

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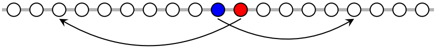

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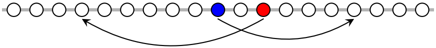

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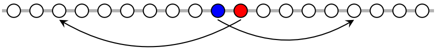

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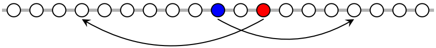

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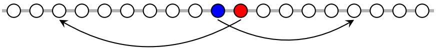

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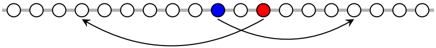

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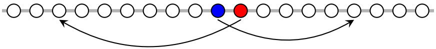

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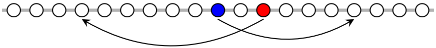

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**General: inductive arguments**

# Further questions

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   For $\mathbf{P}[T_{B_1,C_1}^{A_1} < t_1, T_{B_2,C_2}^{A_2} < t_2] = \mathbf{P}[T_{B_1,C_1}^{A_1} < t_1, T_{B_2,C_2}^{A_2'} < t_2]$, need
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2. Scaling limit of the colored TASEP?

   Two families of TASEPs: LPP and colored TASEP.
   LPP $\to$ Airy Sheet
   $(x, y) \mapsto n^{-1/3}(L_{(xn^{2/3}, -xn^{2/3}),(n-yn^{2/3}, n+yn^{2/3})} - 4n)$
   Colored TASEP?
   $(x, y) \mapsto n^{-1/3}(T^{n^{2/3}x}_{n+n^{2/3}(y-x), n-n^{2/3}(y-x)} - 4n)$?

Thank you!

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