Random Lozenge tiling at cusp and the Pearcey process

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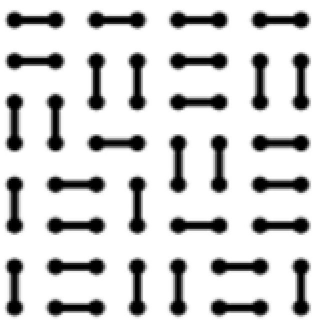
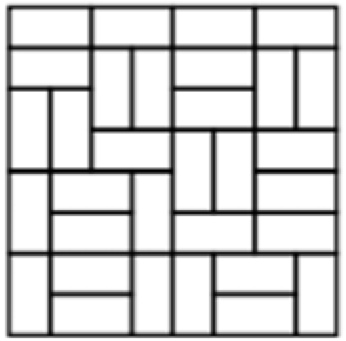
Random Tiling/Dimer Model

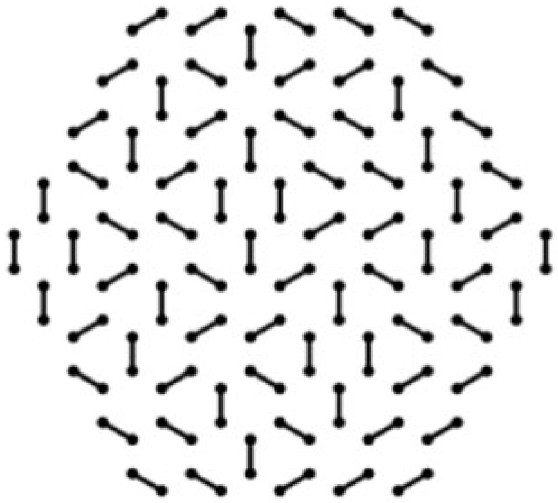
Dimer definition: uniformly chosen perfect matching of a graph.
(covering by edges)

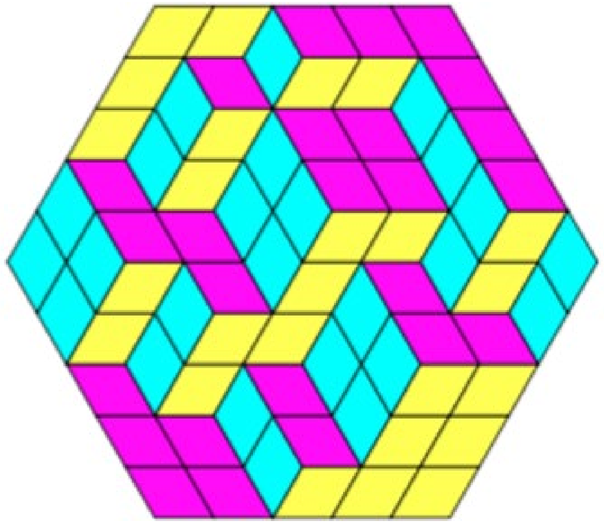
Square lattice: domino tiling



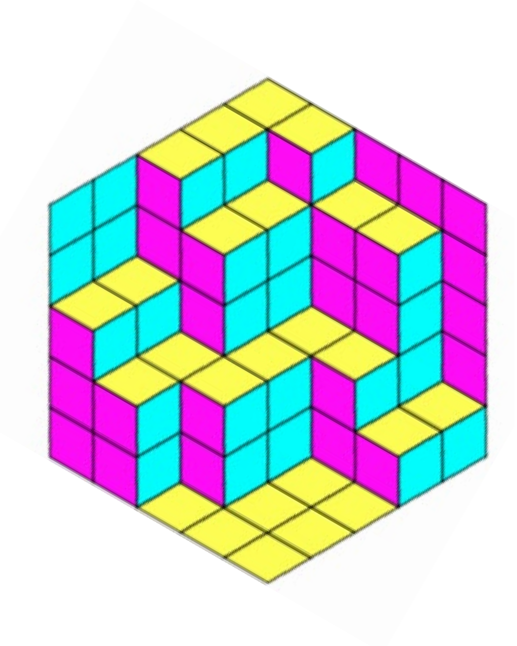


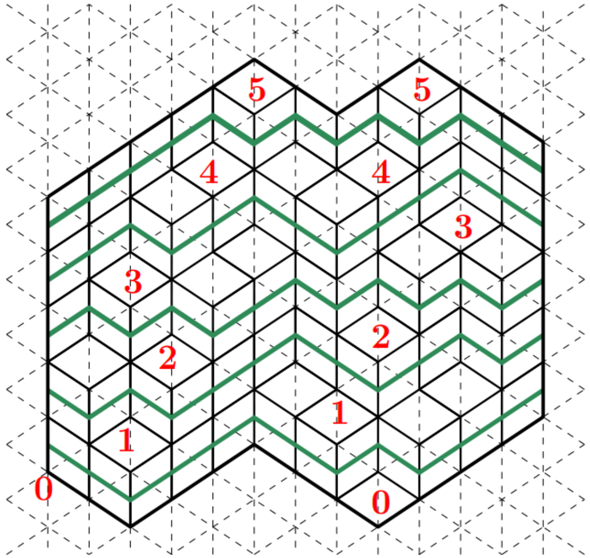
Honeycomb lattice: lozenge tiling

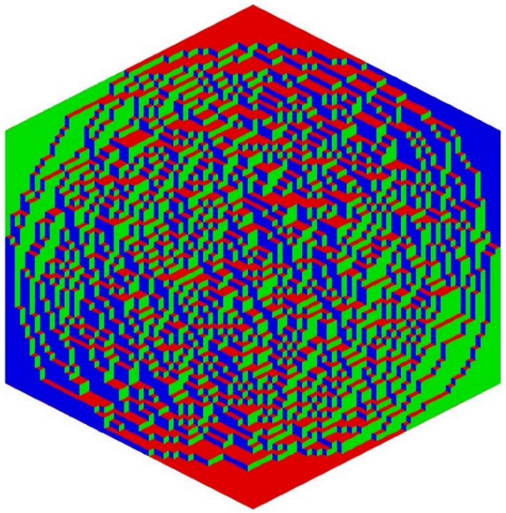


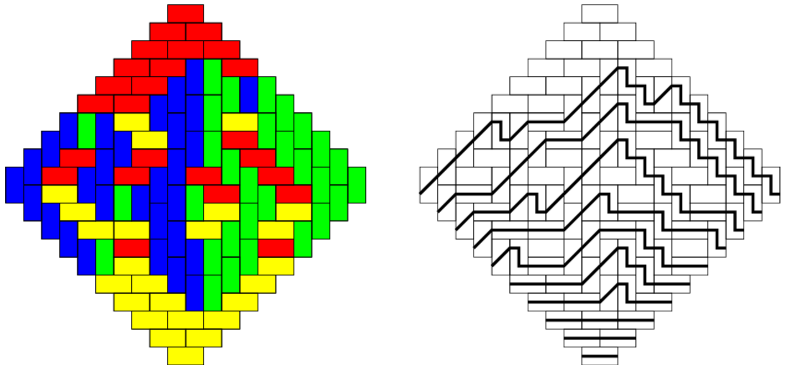


Random Tiling/Dimer Model









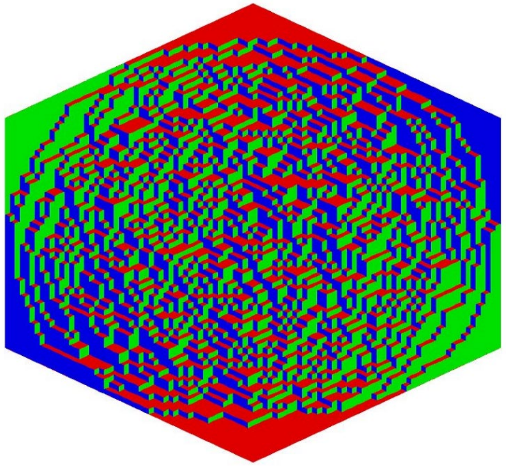
Also for domino tiling

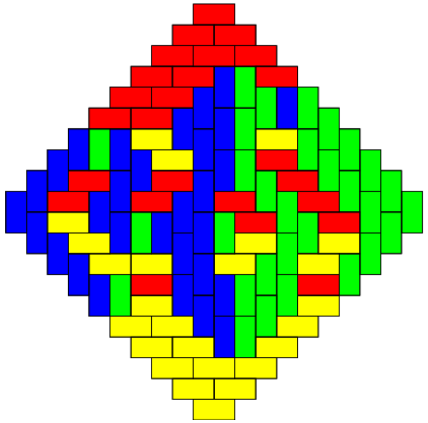
3D visualization: a collection of boxes

- Height function, then a random surface

For a tilable domain, the height function on boundary is determined.

Some motivations





- Natural and beautiful!
- Random surface: a toy model for 3D Ising (zero-temperature limit)
- Bijection with six-vertex (square ice) model (with certain parameters)

Primary interest: large scale behavior?

Law of large number:

(Cohn-Kenyon-Propp, 00) Consider a sequence of tilable domains $R_1, R_2, \ldots$ such that R_n/n converges to a simply connected set Ω (with piecewise smooth boundary), and the boundary height function has scaling limit $h: \partial\Omega \to \mathbb{R}$.
Then for uniform random tiling, the rescaled height function $(x, y) \mapsto H_n(nx, ny)/n$ converges in probability to a **deterministic** function $H^*: \Omega \to \mathbb{R}$.

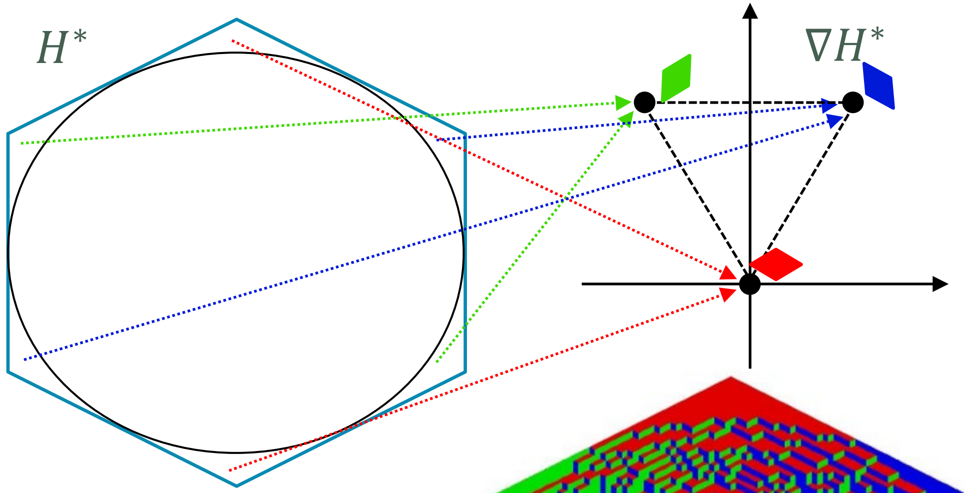
H^* is given by a variational formula (determined by Ω and h).

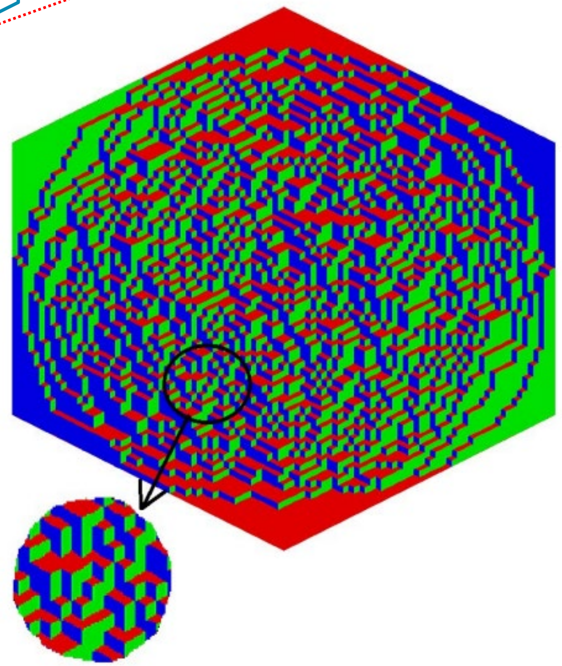
Primary interest: large scale behavior?

Law of large number:

(Cohn-Kenyon-Propp, 00) … the rescaled height function $H_n(nx, ny)/n$ converges to a **deterministic** function $H^*\colon \Omega \to \mathbb{R}$.

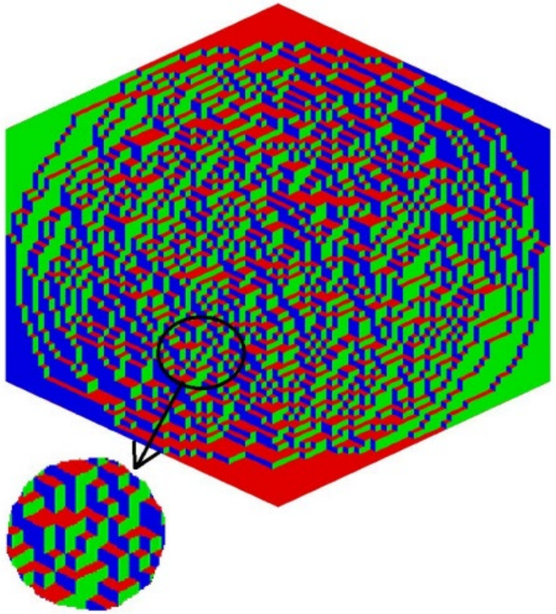
∇H^* describes the slope, corresponding to the 'densities' of each type.





Liquid regions vs frozen regions

Next: fluctuation?



- Global fluctuation: $(x, y) \mapsto H_n(nx, ny) - nH^*(x, y)$

Converges to ***Gaussian Free Field*** in liquid region

Predicted by Kenyon-Okounkov, 05'. The most general setting remains open.

For various domains: Kenyon, 00'; Borodin-Ferrari, 08'; Petrov, 13'; Berestycki-Laslier-Ray, 16'; Bufetov-Gorin, 17'; Chelkak-Laslier-Russkikh, 20'; Huang, 20'; …

- Local fluctuation: $H_n(nx + \cdot, ny + \cdot) - H_n(nx, ny)$: depends on (x, y)

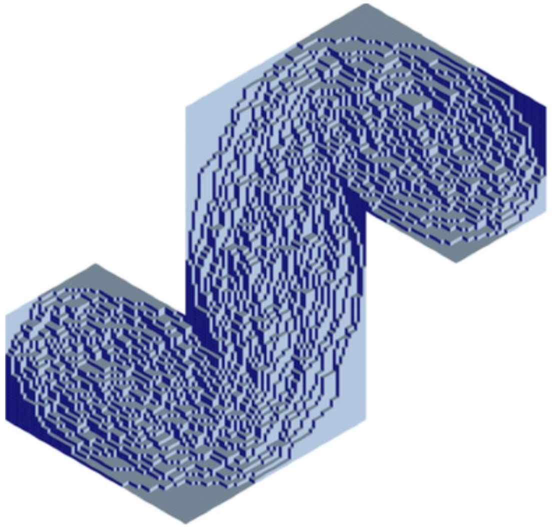
(x, y) in frozen region: just one type

(x, y) in liquid region: $H_n(nx + \cdot, ny + \cdot) - H_n(nx, ny)$ *converges to a* ***translation invariant random function*** *(determined by $\nabla H^*(x, y)$)*

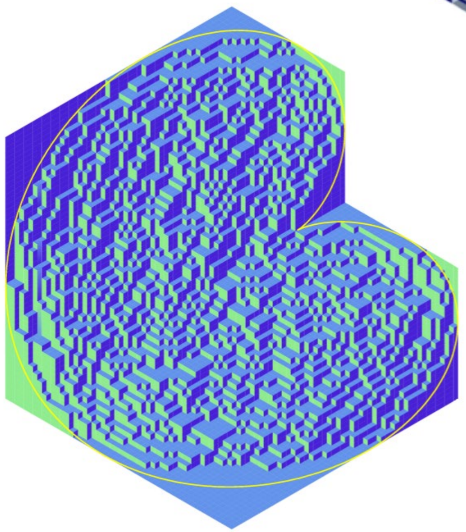
Special domains: Kenyon, 00'; Okounkov-Reshetikhin, 03'; Borodin-Kuan, 10'; Borodin-Gorin-Rains, 10'; Petrov, 14'; Chhita-Johansson, 16; Gorin, 17'; …

Unversality (general domain): Aggarwal, 19'

Arctic curve

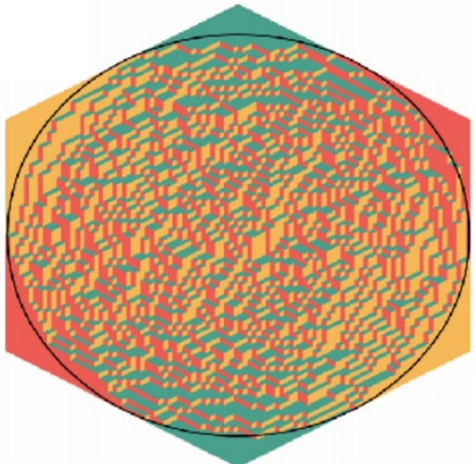


Arctic curve: boundary between liquid and frozen



From now, we consider ***polygonal*** domains

Arctic curve is algebraic for polygonal domains (using complex Burgers equation)
(Kenyon-Okounkov, 05’; Astala-Duse-Prause-Zhong, 20’)

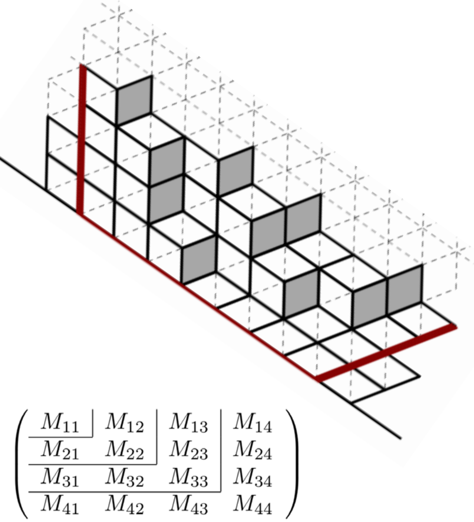


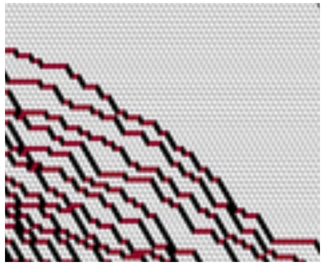
Fluctuation around arctic curve

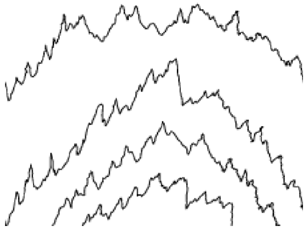
(Two other types for **non-generic** polygons: Airy-cusp and tacnode)

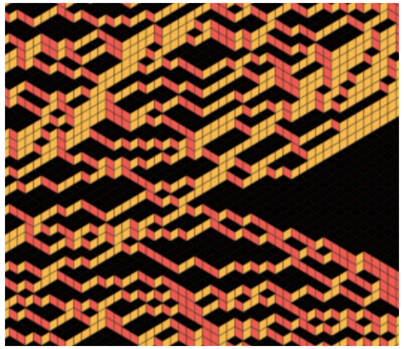
For a generic polygonal domain, around its arctic curve:

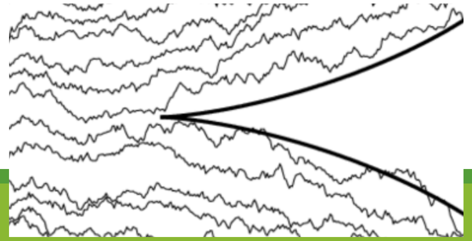
- Airy line ensemble at a smooth point $n^{2/3} \times n^{1/3}$
- Pearcey process at a cusp point $n^{1/2} \times n^{1/4}$
- GUE point process at a tangent point $n^{1/2} \times 1$

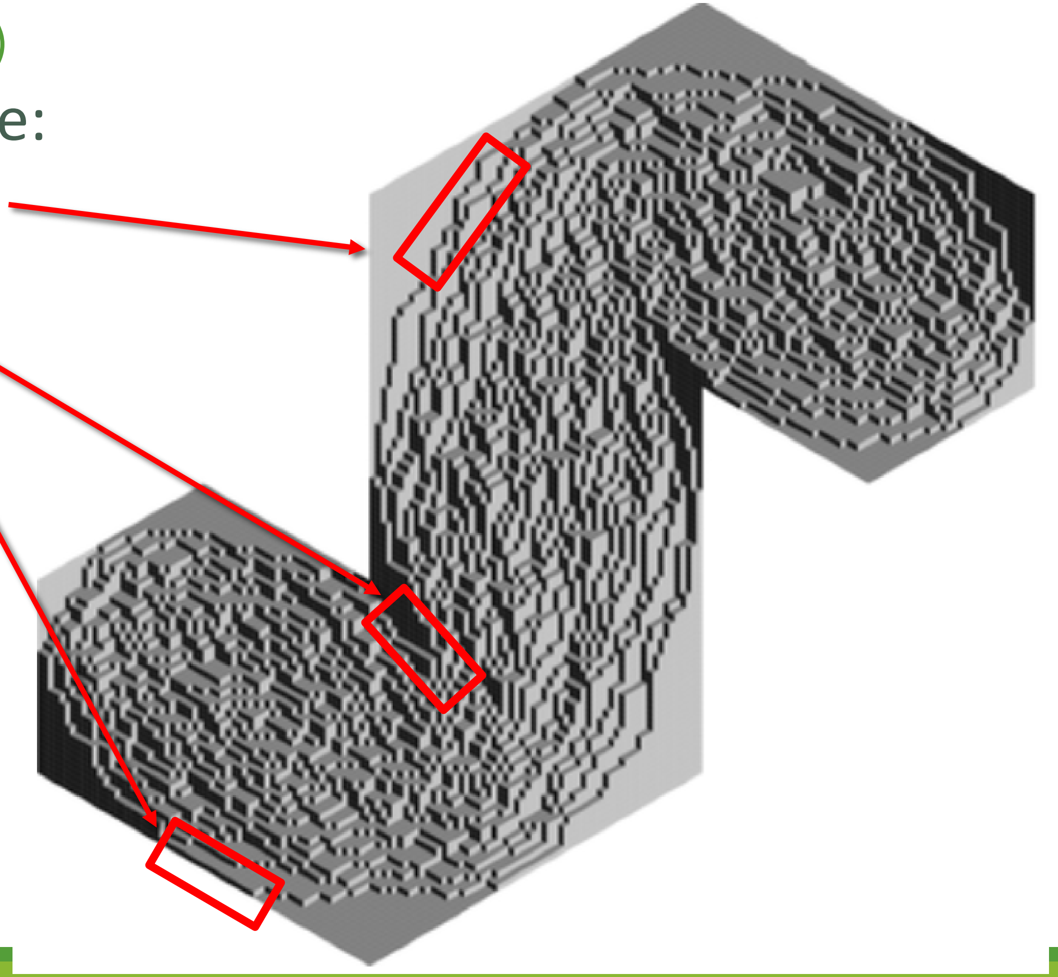








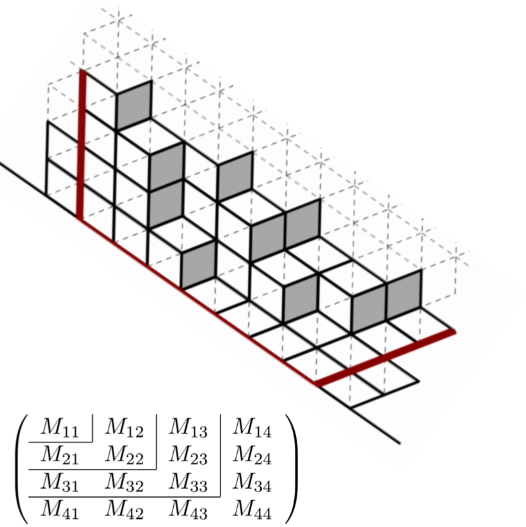


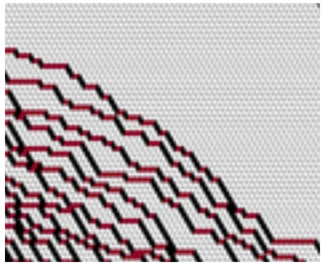


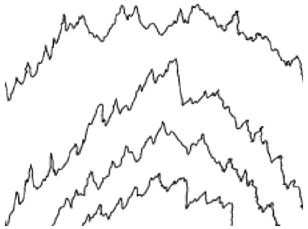
Fluctuation around arctic curve

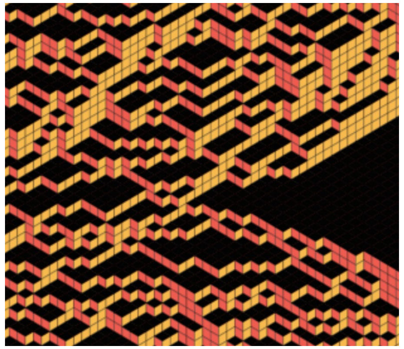
For a generic polygonal domain, around its arctic curve:

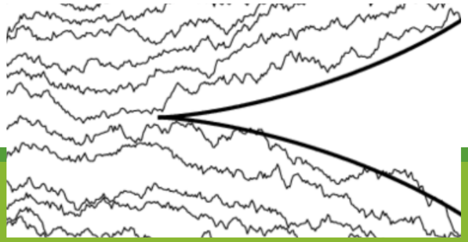
- Airy line ensemble at a smooth point **Universality proved in Aggarwal-Huang, 21'**
- Pearcey process at a cusp point **Today: universality, Huang-Yang-Z., 23'**
- GUE point process at a tangent point **Universality proved in Aggarwal-Gorin, 21'**











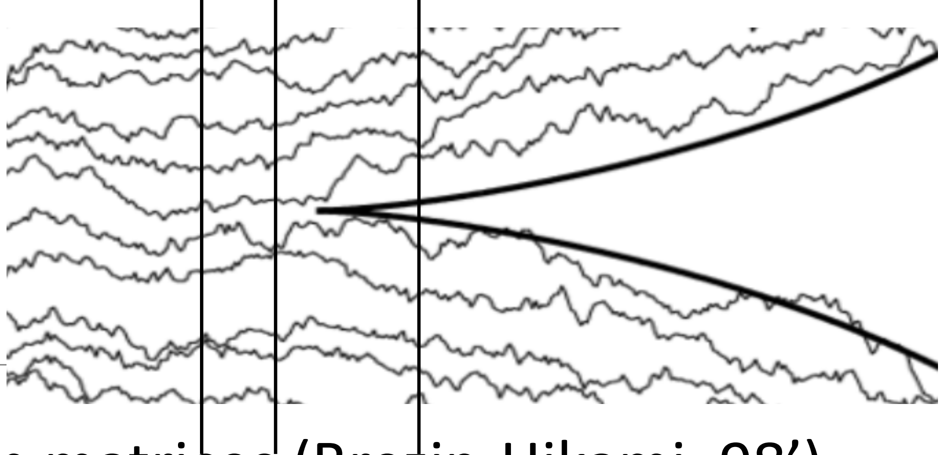
First proved for special domains (hexagon, trapezoid, …)

Universality was then widely predicted

For Pearcey at cusp:

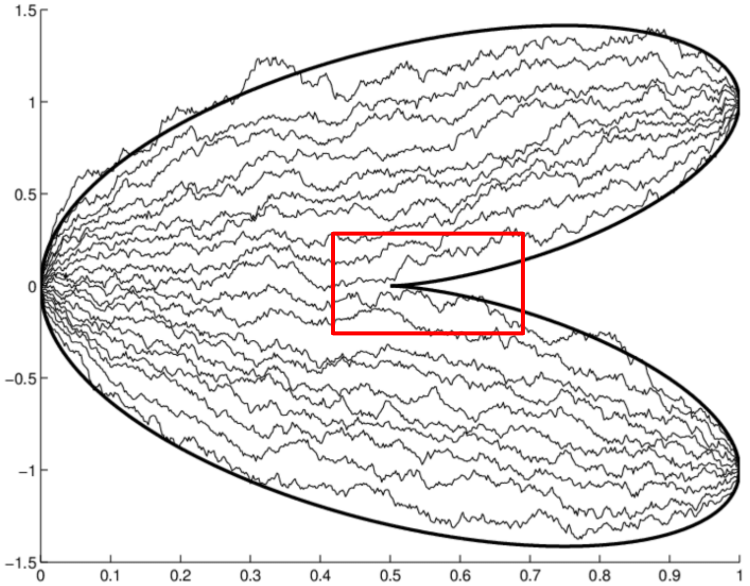
Okounkov-Reshetikhin, 05'; Duse-Johansson-Metcalfe, 15'; Adler-Johansson-van Moerbeke, 16'; Astala-Duse-Prause-Zhong, 20'; Gorin, 21' (*Lectures on random lozenge tilings*)…

Pearcey process

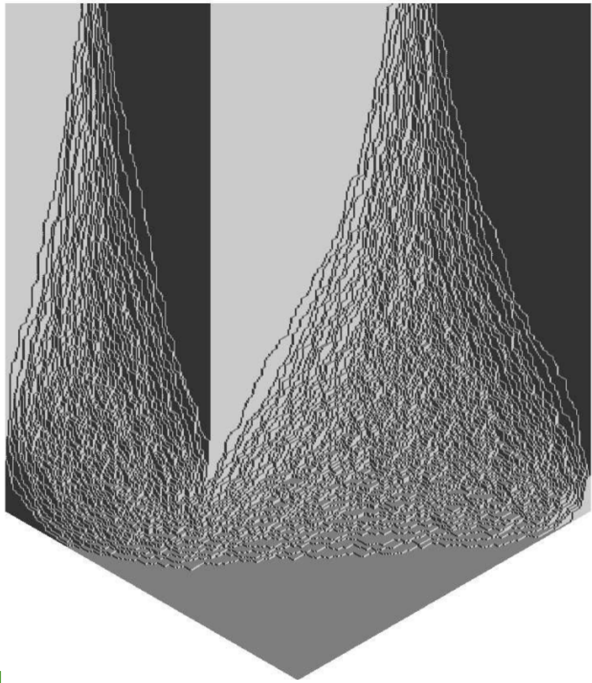


One vertical slice describes eigenvalues of random matrices (Brezin-Hikami, 98’)

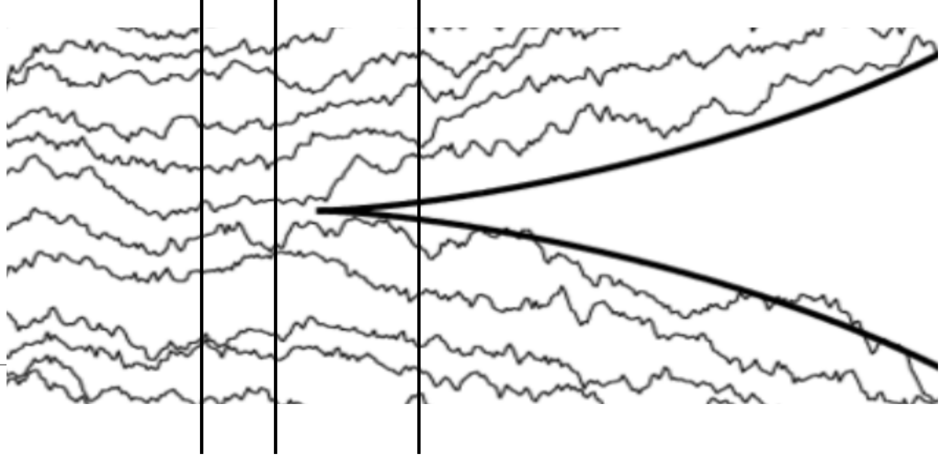
Tracy-Widom, 04’: scaling limit of non-intersecting Brownian bridges



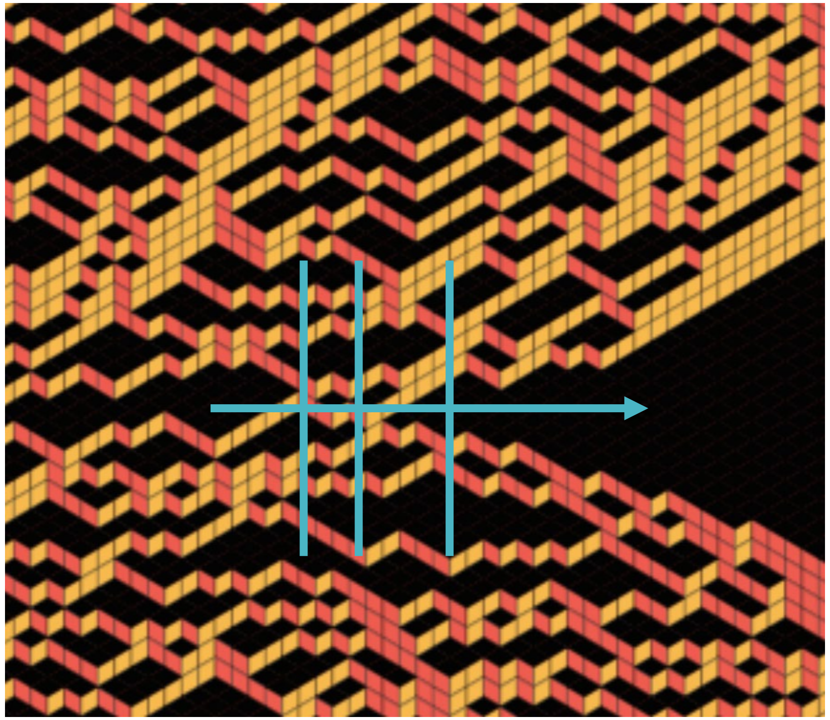
Okounkov-Reshetikhin, 05’:
tiling in a special infinite domain



Pearcey process



Main result (Huang-Yang-Z., 23')



For any generic simply connected polygonal domain, around any cusp point of its arctic curve, the associated paths (under $n^{1/2} \times n^{1/4}$ scaling) converge to the Pearcey process, in the sense of point processes.

(Can be upgraded to uniform convergence)

Proof strategy

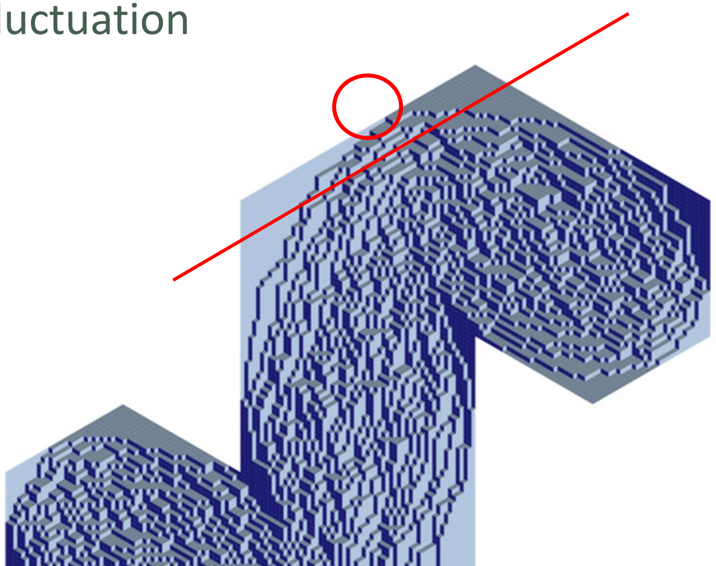
High level idea: compare with known special settings

More 'interior', more subtle

Tangent point: cut a trapezoid

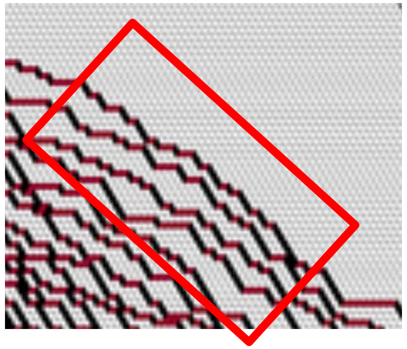
(Aggarwal-Gorin, 21')

Need:
Boundary fluctuation
is $o(n^{1/2})$



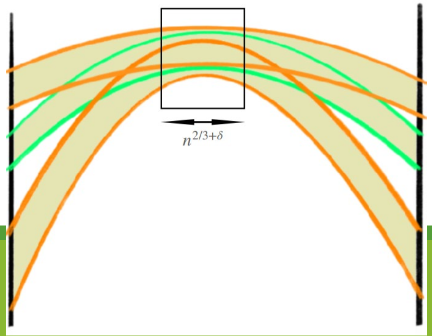
Smooth point: take a box

(Aggarwal-Huang, 21')

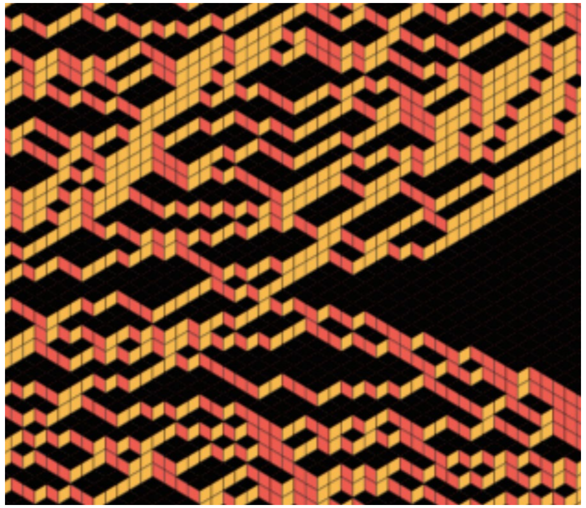


Compare with
Hexagon,
use monotonicity
(sandwich
between two)

Hope:
Boundary
fluctuation is
$o(n^{1/3})$;
Not true!



Cusp point



More 'interior':
fluctuation even grows!
No sandwiching argument

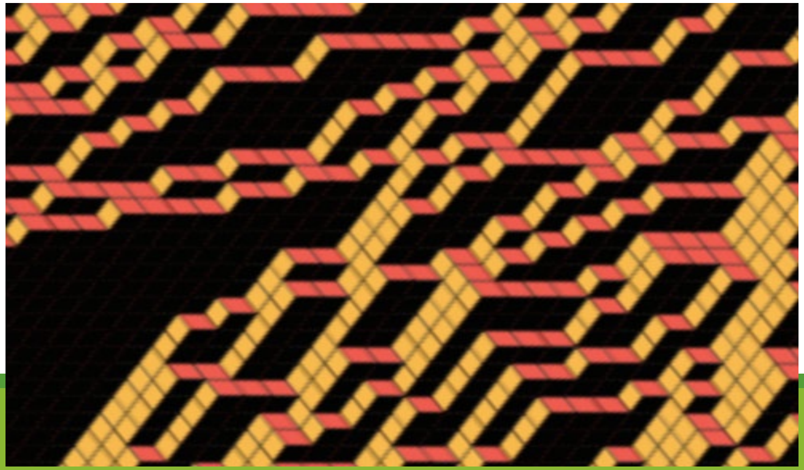
Cusp universality: main steps

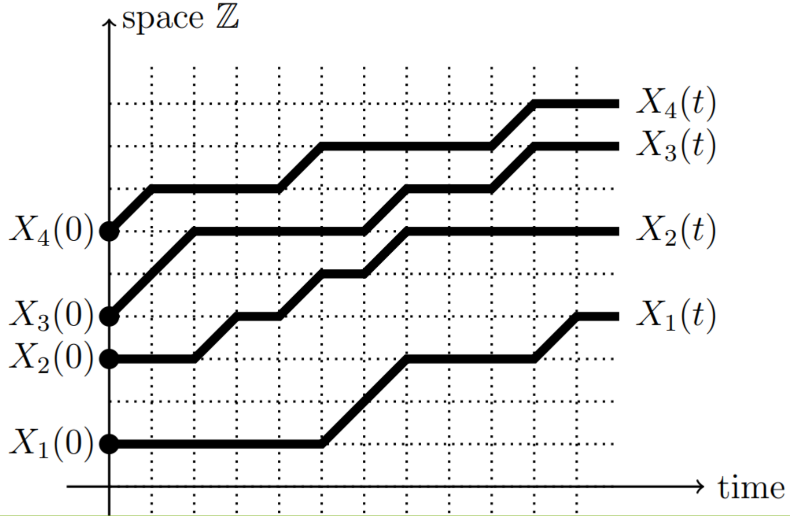
Compare with **non-intersecting Bernoulli random walks (NBRW)**

- Bernoulli(β) random walks conditional on non-intersect up to time ∞
- Markov chain with transition probability

$$\mathbb{P}[X(\mathsf{t}+1)=(\mathsf{y}_{-M},\ldots,\mathsf{y}_N)\mid X(\mathsf{t})=(\mathsf{x}_{-M},\ldots,\mathsf{x}_N)]$$
$$=(1-\beta)^{M+N+1}\prod_{-M\le i\le N}\left(\frac{\beta}{1-\beta}\right)^{\mathsf{y}_i-\mathsf{x}_i}\prod_{-M\le i<j\le N}\frac{(\mathsf{y}_i-\mathsf{y}_j)}{(\mathsf{x}_i-\mathsf{x}_j)}$$

A special case of lozenge tiling: 'free' from top/bottom/right





More tractable formulas

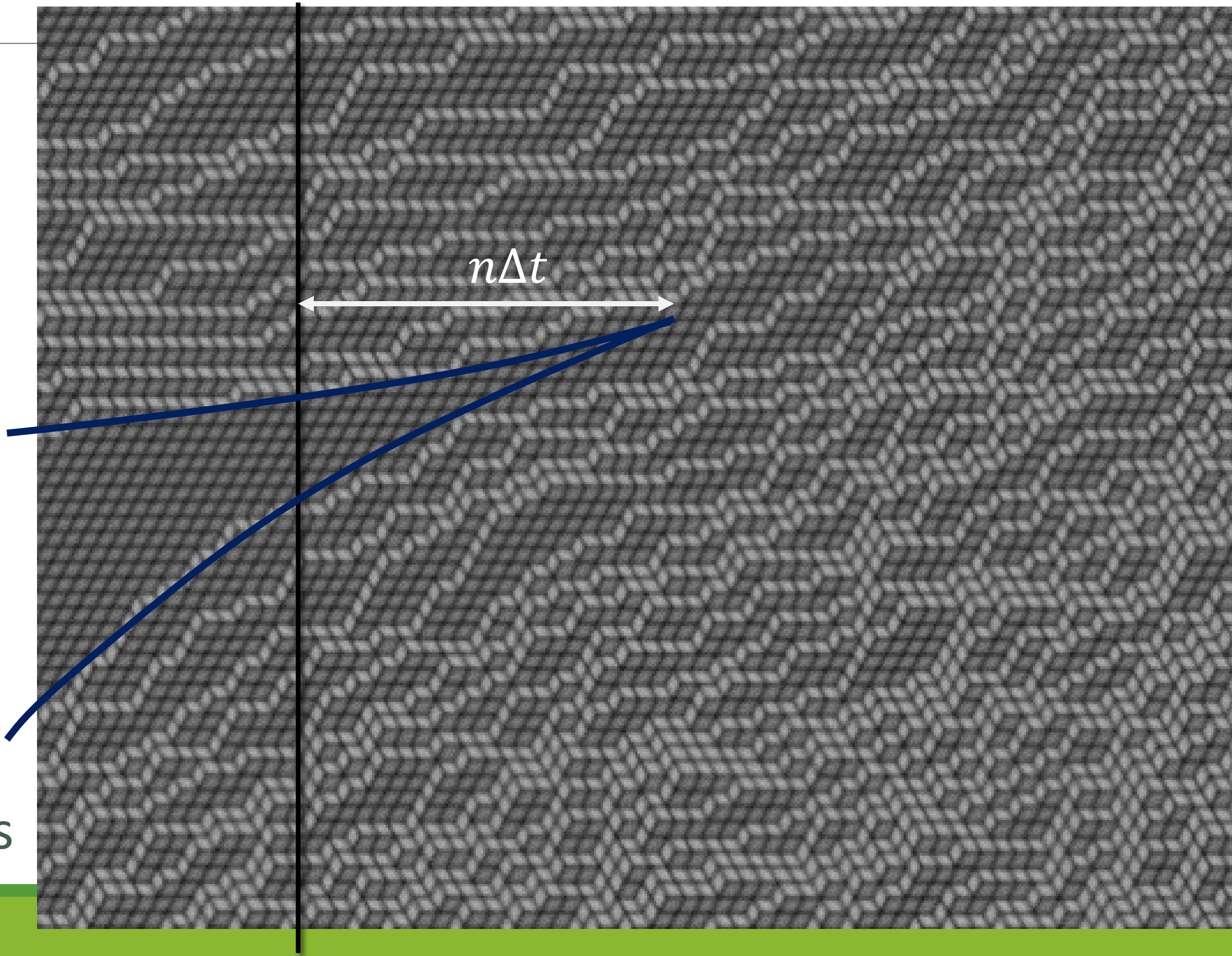
(Petrov, 12'; Gorin-Petrov, 17'; …)

Cusp universality: main steps

The comparison:

- Take the slice at distance $n\Delta t$ from cusp
- Consider NBRW from this slice

(slope parameter β to be determined)



Step 1. (Almost) optimal rigidity for both
(deduced from Huang 21';Aggarwal-Huang, 21')

Step 2. $o(n^{1/4})$ close in expectation

Step 3. NBRW from any 'typical' boundary gives the ***same*** Pearcey process

Step 1. (Almost) optimal rigidity

For each $x_i(t)$, the 'gap' around is
$\sim n^{-1}\partial_x H^*(-t, x_i(t)/n)^{-1}$
(Deduced from Huang, 21'; Aggarwal-Huang, 21')
With high probability,

$$|x_i(tn) - n\gamma_i(t)| < n^{\epsilon}\partial_x H^*(t, \gamma_i(t))^{-1}$$

for each t and i. ($\gamma_i(t)$ is the i-th quantile)

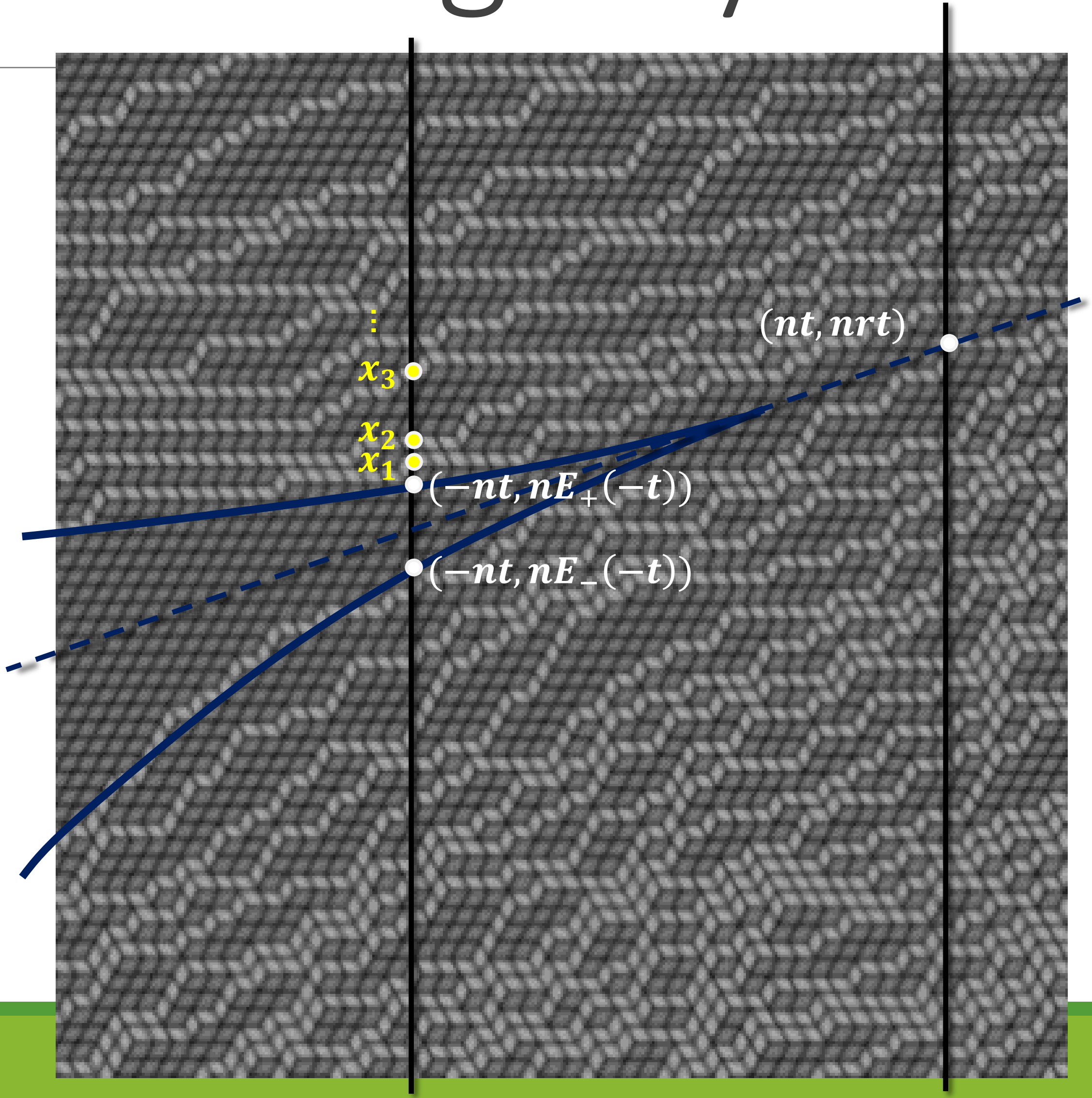
In particular (to the left of cusp)

$$|x_i(-tn) - n\gamma_i(-t)| < n^{1/4+\epsilon}|i|^{-1/4}, \qquad |i| > t^2 n$$
$$|x_i(-tn) - n\gamma_i(-t)| < n^{1/3+\epsilon}t^{1/6}|i|^{-1/3}, \qquad |i| < t^2 n$$

(to the right of cusp)

$$|x_i(tn) - n\gamma_i(t)| < n^{1/4+\epsilon}|i|^{-1/4}, \qquad |i| > t^2 n$$
$$|x_i(tn) - n\gamma_i(t)| < n^{\epsilon}t^{-1/2}, \qquad |i| < t^2 n$$

Same for NBRW
(but potentially **different** cusp location and γ_i!)



Step 2. Compare deterministic part

Use Burger's equation (but extend to complex plane; Kenyon-Okounkov, 05')

$$\partial_t f + \partial_z f \frac{f}{f+1} = 0$$

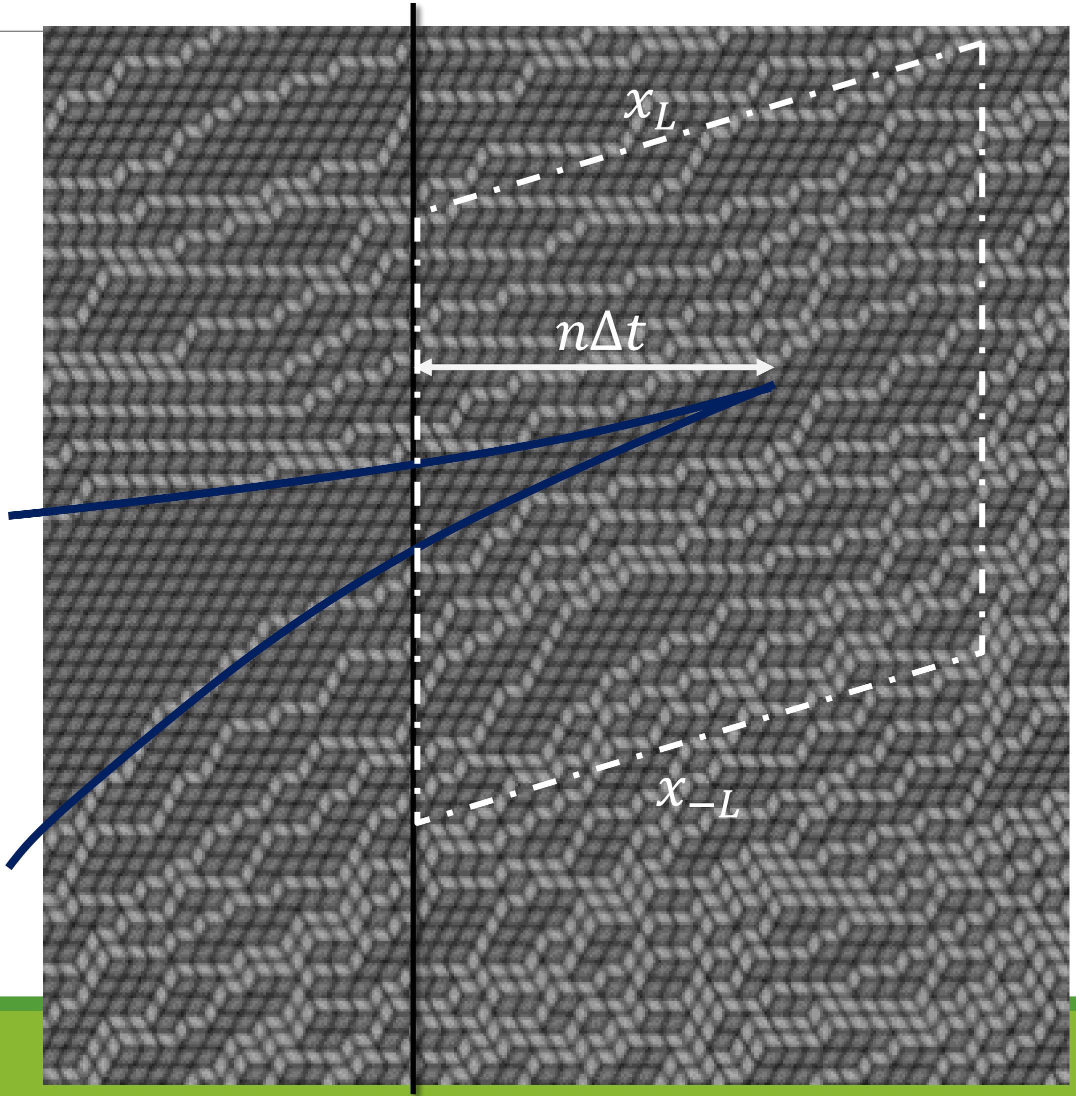
Reduced to comparing f with different initial conditions (evolving for time Δt)

- Cusp locations: distance $< n(\Delta t)^2$
- Upper/lower boundary:

  $$|n\gamma_L(t) - n\gamma_L{}'(t)| < n^{1+\epsilon}(\Delta t)^{5/2} \,,$$

  for $L = n^{1+2\epsilon}(\Delta t)^2$, $|t| < \Delta t$
- Right boundary: for $|i| < L$,

  $$|n\gamma_i(\Delta t) - n\gamma_i{}'(\Delta t)| < n^{1+\epsilon}(\Delta t)^2 \,.$$



Comparison (tiling vs NBRW):
deterministic + fluctuation

$L = n^{1+2\epsilon}(\Delta t)^2$

Deterministic:

Upper/lower/right boundary expectation differ by $n^{1+\epsilon}(\Delta t)^2$

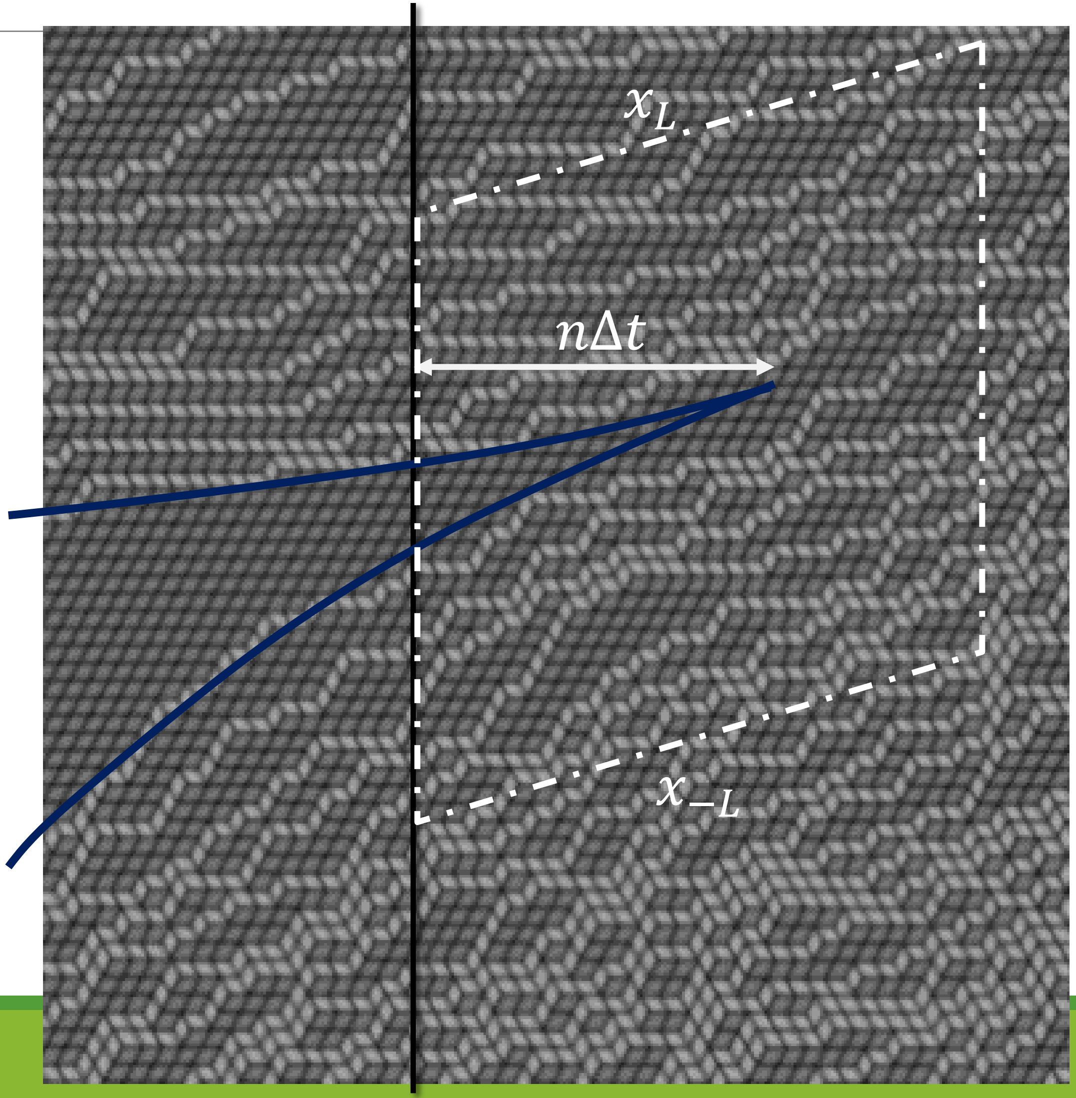
Fluctuation:

Upper/lower fluctuates by $< n^{1/4+\epsilon} L^{-1/4} = n^{-\epsilon/2}(\Delta t)^{-1/2}$

Right fluctuates by $< n^{\epsilon} (\Delta t)^{-1/2}$

Can take $\Delta t = n^{-0.49}$, then all $\ll n^{1/4}$

➢Tiling and NBRW are the *same*



Step 3. Cusp universality for NBRW

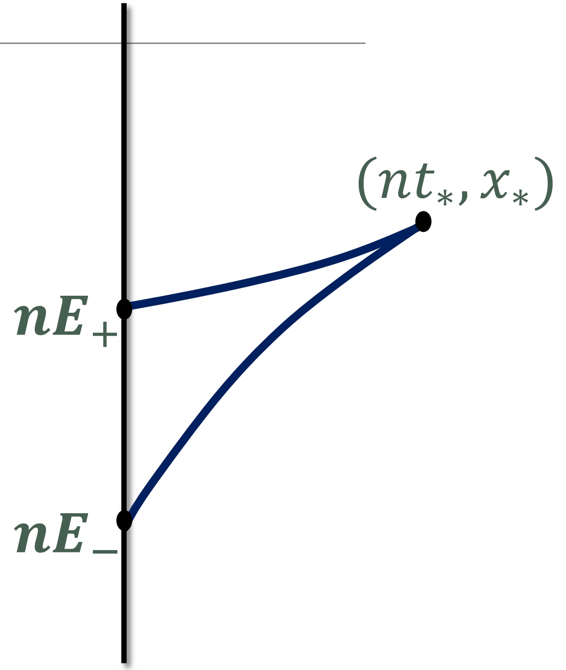
Consider any NBRW with initial data $\{x_i\}_{i=-M}^{N}$, such that for some $n^{-1/2+\epsilon} < t_0 < n^{-\epsilon}$, and $E_+ - E_- \sim t_0^{3/2}$,

$$x_i - nE_+ \sim t_0^{1/6} n^{1/3} i^{2/3}, \qquad nE_- - x_{-i} \sim t_0^{1/6} n^{1/3} i^{2/3}$$

when $i < t^2 n$,

$$x_i - nE_+ \sim n^{1/4} i^{3/4}, \qquad nE_- - x_{-i} \sim n^{1/4} i^{3/4}$$

when $i > t^2 n$.



Then can find x_* and $t_* \sim t_0$, and p, q, r, such that around (nt_*, x_*), with scale $pn^{1/2}$ and $qn^{1/4}$, and slope r, there is ‘roughly’ Pearcey process.

Asymptotic analysis for formulas of NBRW from Gorin-Petrov, 16’; *steepest descent method*

Special case done in Okounkov-Reshetikhin, 05’

This is a ‘small-distance’ result ($nt_* < n^{1-\epsilon}$) and is subtle

Summary and further comments

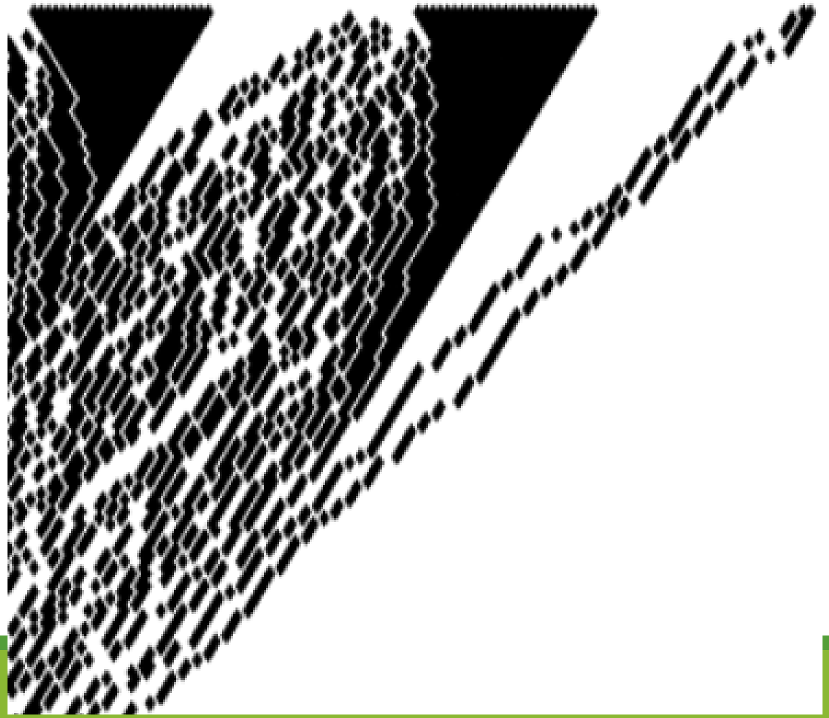
For lozenge tiling in a generic simply connected polygonal domain, we prove cusp universality of the Pearcey process, by

- carefully comparing tiling and NBRW (using optimal rigidity from Huang, 21'; Aggarwal-Huang 21' as an input)
- deriving a small-scale cusp universality for NBRW (doing refined asymptotic analysis for formulas)

Beyond polygon?

Can be subtle: sensitive to microscopic boundary perturbation

How boundary perturbation affects scaling?



Thank you!

Some figures are from Petrov’s website.
(https://lpetrov.cc/2016/08/Tilings-examples-inline/)
and the textbook *Lectures on random lozenge tilings* by Gorin